A packing model to estimate VANET capacity

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***Abstract*—In IEEE 802.11p VANET networks the transmission scheduling is distributed and asynchronous. The number of simultaneous transmitters is thus closely related to the CSMA/CA mechanism which limits the spatial reuse of the channel. The capacity is bounded by a constant C whatever the number of nodes and the type of routing schemes. This paper aims to evaluate the spatial reuse of a VANET (using CSMA/CA) and to deduce its maximum capacity. The proposed model is an extension of a classical *packing problem*. We prove formally that the intensity of the maximum number of simultaneous transmitters (the maximal spatial reuse) converges to a constant and propose a simple estimate of this constant. Realistic simulations show that the theoretical capacity offers a very tight bound on the real capacity of the network.**

I. Introduction

In recent years, Inter-Vehicle Communication (IVC) has become an intense research area, as part of Intelligent Transportation Systems. It assumes that all or a subset of the vehicles is equipped with radio devices, enabling communication between them. Although classical 802.11 can be used for IVC, specific technologies such as IEEE 802.11p [1] (also referred to as Wireless Access in Vehicular Environments, WAVE) have been standardized to support these communications. This standard includes data exchanges between vehicles (ad hoc mode) and between infrastructure and vehicles. When the ad hoc mode is used, the network formed by the vehicles is called a Vehicular Ad hoc NETwork (VANET).

VANET has been designed for two families of applications. The first family corresponds to applications directly used by the users, enabling voice or chat communications between vehicles, advertising gas stations, restaurants, traffic conditions, etc. But the most important applications are related to road safety. Information on road conditions, speed, traffic or alert messages (signaling an accident) may be exchanged in the VANET allowing drivers to anticipate dangerous situations [2]. Data from embedded sensors may also be exchanged in order to increase the perception of the environment. This helps drivers to make appropriate decisions, as it increases the information available on road conditions and traffic situations. But the network capacity, i.e. the amount of data that the network is able to carry is limited. Applications cannot rely on an infinite capacity and cannot suppose that all data can be exchanged at any rate. For instance, advanced driver assistance systems must be designed in function of the available capacity, i.e. select the most pertinent information and the frequency of their exchanged so that application requirements does not exceed the capacity. A rigorous evaluation of VANET capacity is thus crucial.

A theoretical bound on the capacity of ad hoc networks was already investigated in [3] where the authors prove that, in a network of n nodes, a capacity of $\Omega\left(\frac{1}{\sqrt{n\cdot\log n}}\right)$ is feasible. This bound has been improved in a lot of studies [4], [5], [6] for different scenarios and radio environments. But all these studies deal with the asymptotic behavior of the capacity with regard to the number of nodes and do not propose precise estimates of this capacity. Moreover, routing and MAC schemes considered in these papers do not correspond to real deployments of ad hoc networks. For example, in most of these studies the medium is shared through a TDMA mechanism whereas most of the ad hoc technologies use CSMA/CA instead (Wi-Fi, Zigbee, 802.11p).

However all these studies focus on networks where nodes are distributed on the plane or in a 2-dimensional observation window. VANETs have very different topologies as the vehicles/nodes are distributed along roads and highways. Radio range of the nodes (about 700 meters with 802.11p in rural environment) being much greater than the road width, we can consider that the topology is distributed on a line rather than in a 2 dimensional space. Lines, grids or topologies composed of a set of lines (to model streets in a city) are thus more appropriate to model VANET topologies. There are only a few studies dealing with the estimation of the capacity in this context. In [7], [8], the authors propose a bound on VANET capacity. They show that when nodes are at constant intervals or exponentially distributed along a line, the capacity is $\Omega\left(\frac{1}{n}\right)$ and $\Omega\left(\frac{1}{n\cdot ln(n)}\right)$ in downtown (city) grids. But it is also an asymptotic bound. Moreover, physical and MAC layers are unrealistic, radio ranges are constant and the same for all the nodes, interference is not taken into account and they assume a perfect transmission scheduling between the nodes. Thus, this bound cannot be applied to 802.11p networks. In [9], the broadcast capacity of a VANET is estimated. The idea is similar to this paper: an estimation of the number of simultaneous transmitters is proposed. But, this evaluation is based on numerical evaluation only, using integer programming.

II. Contribution and organization

In CSMA/CA based wireless networks, the transmission scheduling is distributed and asynchronous. The number of simultaneous transmitters is thus closely related to the CSMA/CA mechanism which limits the spatial reuse of the channel. The total number of frames sent in the whole network, and thus the capacity, is bounded by a constant C whatever the number of nodes and the type of routing schemes. This constant has been evaluated in [10] for 2-dimensional networks but not for VANET. This paper aims to evaluate the spatial reuse of a VANET (using CSMA/CA) and to deduce its maximum capacity. We propose an extension of the *packing problem* proposed by Renyi [11]. This model mimics the CSMA/CA procedure used in the IEEE 802.11p technology. We prove formally that the intensity of the maximum number of simultaneous transmitters (the maximal spatial reuse) converges to a constant. This quantity corrresponds to the maximum number of simultaneous transmitters per unit length (per kilometer in our simulations). Also, we propose simple estimates of this constant. Finally, we give formula which links this constant to the VANET capacity.

In order to validate the theoretical results and the different estimations, we performed a large number of simulations. We interfaced the NS-3 [12] network simulator with a traffic simulator emulating vehicles' trajectory on a highway. Combination of the vehicle traffic generator and NS-3 (for the MAC, IP and application layer) allowed us to perform simulations as realistic as possible and to validate our model. Simulations show that the proposed theoritical model offers a very accurate bound on the capacity of VANET and can thus be used as a dimensioning or parametric tool for applications.

The paper is organized as follows. In Section III we present the technological context of this study. The theoretical model is presented in Section IV. Simulations and analytical results are compared in Section V. We conclude in Section VI.

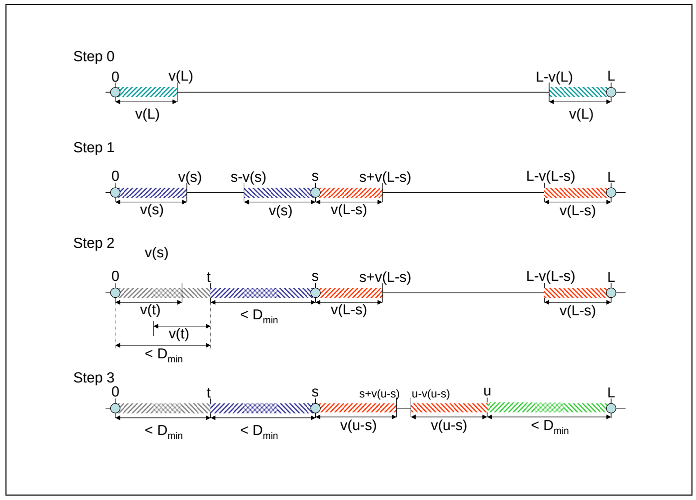


Fig. 1. A sample of our model. Step 0 (initialization): two nodes are located at 0 and L. Step 1: a new point is uniformly distributed in $[v(L), L-v(L)]$, at s in our example. There are two intervals where transmitters can be added : $[v(s), s-v(s)]$ and $[s+v(L-s), L-v(L-s)]$. Step 2: a new point is uniformly distributed in $[v(s), s-v(s)]$. It is located at t. Interval on the left and right of t are smaller than D_{min}. Therefore, points cannot be added in these two intervals. Step 3: a new point u is uniformly distributed in $[s+v(L-s), L-v(L-s)]$. The interval on the right hand side of u is smaller than D_{min}. But a new point can be added on the left, in the interval $[s+v(u-s), u-v(u-s)]$. This is done at step 4 (not shown in the figure). This terminates the process.

III. 802.11P MAC OVERVIEW

The IEEE 802.11p spectrum is based on DSRC which is composed of six service channels and one control channel. The control channel will be used for broadcast communications dedicated to high priority data and management frames, especially for safety communications. It should be the privileged channel used to disseminate messages from safety applications. The service channels can be used for safety and service applications, broadcast and unicast communications. The MAC layer in 802.11p is similar to the IEEE 802.11e Quality of Service extension. Application messages are categorized into one of four different queues depending on their level of priority. Each queue uses the classical CSMA/CA (Carrier Sense Multiple Access/Congestion Avoidance) mechanism to access the medium, but CSMA/CA parameters (backoff, etc.) are different from one queue to another in order to favour frames with high priority. In CSMA/CA, a candidate transmitter senses the channel before effectively transmitting. Depending on the channel state, idle or busy, the transmission is started or postponed. *Clear Channel Assessment* (CCA) depends on the MAC protocol and the terminal settings. For the CSMA/CA protocols used in IEEE 802.11, CCA is performed according to one of these three methods.

1) CCA Mode 1: *Energy above threshold*. CCA shall report a busy medium upon detecting any energy above the Energy Detection (ED) threshold. In this case, the channel occupancy is related to the total interference level.
2) CCA Mode 2: *Carrier sense only*. CCA shall report a busy medium only upon the detection of a signal compliant with its own standard, i.e. same physical layer (PHY) characteristics, such as modulation or spreading. Note that depending on threshold values, this signal may be above or below the ED threshold.
3) CCA Mode 3: *Carrier sense with energy above threshold*. CCA shall report a busy medium using a logical combination (e.g. AND or OR) of Detection of a compliant signal AND/OR Energy above the ED threshold.

Therefore, the CCA mechanism ensures that there is a minimal distance between simultaneous transmitters (except in case of a collision), it also limits the total number of simultaneous transmitters over the line and thus the number of frame which can be sent per second. Hence, the problem of determining capacity of a VANET in highway scenarios turns out to determine the number of simultaneous transmitters over a line.

IV. MODEL

a) Assumptions: The proposed model mimics the CCA mode 1, where the sum of signals from all transmitters is taken into account to detect the medium idle or busy. With this mode, a node will be allowed to transmit its frame if the measured interfence is lower than a pre-defined threshold θ. We consider a path-loss function $l(.)$, which gives the reception power of a signal as function of the distance from the transmitter.

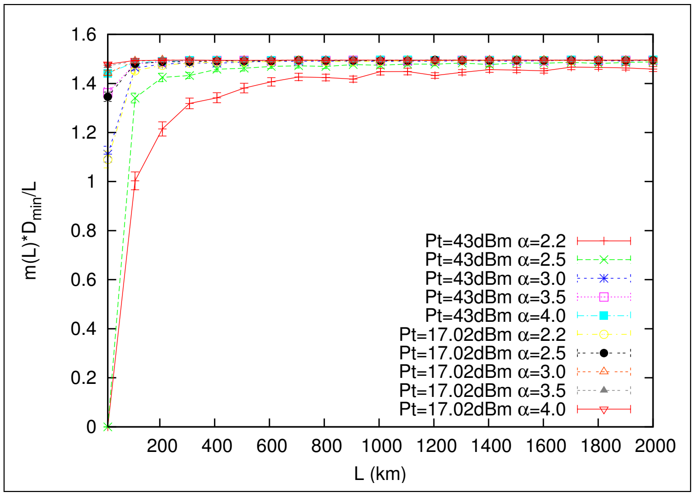


Fig. 2. Convergence of $\frac{m(L)D_{min}}{L}$ as L increases for $l(u) = Ptmin(B, \frac{B}{u^{\alpha}})$ and different value of α and P_t. D_min is the solution of $2l(\frac{D_{min}}{2}) = \theta$ with $\theta = -99dBm$.

We assume that $l(.)$, defined in $\mathbb{R}^+$, is positive, continuous, decreasing with $l(0) > \theta$ (θ is the CCA threshold) and $\lim_{u\to+\infty} l(u) = 0$. Also, we assume that interference $I(x)$ at x ($x \in \mathbb{R}^+$) is generated by the two closest transmitters:

$$I(x) = l(x - Le) + l(Ri - x) \quad (1)$$

where Le, Ri are the two closest transmitting nodes around x, the closest one on the left (located at Le) and on the right (located at Ri). This assumption does not introduce a bias in the results since the 802.11p technology have a great radio range (up to 1km according to the standard). Hence, only the closest transmitters may generate significant interference. According to CCA mode 1, a node at x can transmit a frame, and be a transmitter, if and only if:

$$I(x) = l(x - Le) + l(Ri - x) < \theta \quad (2)$$

Between two successive transmitters there is a sub-interval where new transmitters can access to the medium. It is represented in Figure 1 (step 0). In this figure, two transmitters are at distance L from each others. Around each transmitter there is an interval where the interference level (sum of the signal from these two transmitters) is above θ. These intervals corresponds to the hatched rectangle in the figure. These intervals are symmetric and depend on the distance between the two interferers on the left and on the right (at 0 and L), the path-loss function and the threshold θ. Their lengths may be described with the following function. Let $v(s)$ with $s \in \mathbb{R}^+$ be a function defined as the solution of:

$$l(v(s)) + l(s - v(s)) = \theta \quad (3)$$

$v(L)$ sets the minimal distance from the current transmitters at which interference is less than θ. The interval where a new transmitter can be added is thus $[L, L - v(L)]$. It makes sense only if L is sufficiently great. The interval is not empty only if $L < D_{min}$ with D_{min} solution of $2 \cdot l(\frac{D_{min}}{2}) = \theta$. The function $v(.)$ is thus defined in $[D_{min}, +\infty]$.

b) Model: The proposed process models locations of the simultaneous transmitters on a highway with length L. The considered interval is thus $[0, L]$. The model aims to represent the maximum number of transmitters in $[0, L]$ such that the CCA rule given by equation (2) is respected.

Formally, the process is built as follows. We assumed that there is two initial transmitters at locations 0 and L. If $L > D_{min}$, a new transmitter is uniformly distributed in $[v(L), L - v(L)]$. Let s be its location. If $s > D_{min}$, a new point is uniformy distributed in $[v(s), s - v(s)]$ and if $L - s > D_{min}$ a new point is uniformly distributed in $[s + v(L-s), L - v(L-s)]$. Each time a new point is added, it creates a new interval on its left and its right. If the length of an interval is less than D_{min} we cannot add a new point, otherwise we add a new point uniformly distributed in this interval. The process stops when all intervals are smaller than D_{min}. An example of this process is represented in Figure 1.

We denote $m(L)$ the mean number of points in the considered interval ($[0, L]$). Unfortunatley, its computation is, to our knowledge, untractable. Nevertheless, we can propose some results about its intensity (mean number of points per unit length).

Proposition 1. *Let $m(L)$ be the mean number of points in the interval $[0, L]$ for the process defined above, then:*

$$\lim_{L\to+\infty} \frac{m(L)}{L} = a \quad (4)$$

where a is positive constant.

The proof is given in appendix. This proposition proves that the intensity of this point process converges to a constant as the size of the interval increases. This constant can be used to evaluate the mean number of transmitters and the capacity of the VANET. Indeed, $m(L)$ can be evaluated as $a \times L$. It corresponds to the mean number of simultaneous transmitters on a road of length L. Consequently, the capacity which is defined as the mean number of frames sent per second in the network can be estimated as:

$$Capacity(L) = \frac{aL}{T} \quad (5)$$

where T is the mean time to transmit a frame. This time takes into account the DIFS, the time to transmit the frame, the SIFS and the acknowledgment.

c) Estimation of a: According to equation 5, estimation of the capacity boils down to the computation of the limit a. We propose an estimation of a which does not require any simulation and can be deduced directly from the path-loss function. In Figure 2, we plotted the quantity $\frac{m(L)D_{min}}{L}$ when L increases. Each point is the average of 100 samples and is shown with a confidence interval at 95%. The considered path loss function is $l(u) = Pt \cdot min(B, \frac{B}{u^{\alpha}})$, where Pt is the transmission power, B is the loss reference parameters (equals to -46.6dBm) and α is the path-loss exponent. In this figure, we took into account two transmitting powers $Pt = 17.02$dBm and $Pt = 43$dBm corresponding to transmission powers used in $802.11a$ and $802.11p$ technologies and different path-loss exponent α modeling different radio environment. We observe that all curves converge to the same constant, approximatively equal to 1.49. This result is not surprising as it holds for other packing problems in one or two-dimensional spaces (see [11] or [10] for instance). We also performed the same simulations for other path-loss function (with exponential decay for example), and observe a convergence to the same constant. These results are not shown here by lack of space. This convergence to a universal constant allows us to estimate the limit a of Proposition 1 as follow:

$$\lim_{L \to +\infty} \frac{m(L)}{L} = a \approx \frac{c}{D_{min}} \tag{6}$$

with $c = 1.49$ and D_{min} solution of $2l(\frac{D_{min}}{2}) = \theta$. The final capacity is then evaluated as:

$$Capacity(L) = \frac{cL}{D_{min}T} \tag{7}$$

V. SIMULATIONS

A. Simulators and parameters

In order to validate the theoretical capacity, we simulated the capacity of a VANET using teh network simulator NS-3. In these simulations, nodes are equipped with IEEE 802.11p interfaces. The parameters are given in Table I. Vehicles are located along a line modeling a highway of 50km. Each point in the different figures are computed as the mean of 100 simulations and are presented with a confidence interval at 95%. Each vehicle is a CBR (Constant Bit Rate) source where the destination is a neighbor. The CBR rate is close to the 802.11p rate in order to saturate the network. The capacity is computed as the total number of frames properly received by the vehicles. The transmitters' intensity is computed as the number of simultaneous transmitters at a given time during the simulation. Also, we recorded the exact locations of these transmitters. They are used to evaluate distribution of the distance between successive transmitters. To avoid edge effects all these measurements are done between the kilometer 5 and 45 km of the highway.

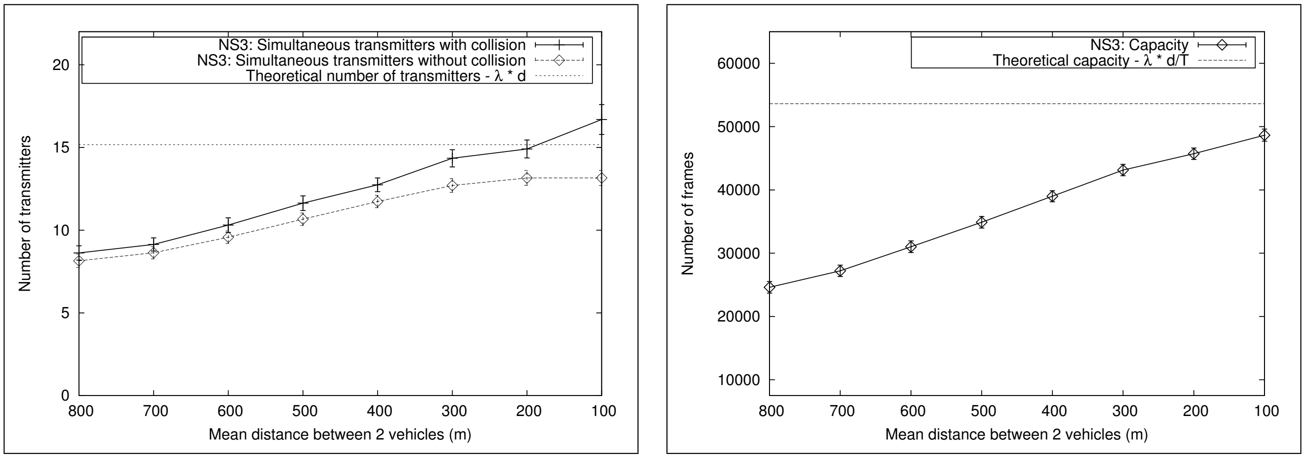
Vehicles' location is set according to an external simulator. This is a micro-simulator emulating driver' behaviors on a highway. This traffic simulator allows us to faithfully emulate driver behavior. We describe this simulator in a few words. On a highway, driver behavior is limited to accelerating, braking and changing lanes. We assume that there is no off-ramp on the section of highway. A desired speed is associated with each vehicle. It corresponds to the speed that the driver would reach if he was alone in his lane. If the driver is alone (the downstream vehicle is sufficiently far), he adapts his acceleration to reach his desired speed (free flow regime). If he is not alone, he adapts his acceleration to the vehicles around (car following regime). He can also change lanes if the conditions of another lane seem better. All these decisions are functions of traffic condition (speed and distance) and random variables used to introduce a different behavior for each vehicle. This kind of simulation is called micro simulation [13], and the model we used is presented in detail in [14]. The model has been tuned and validated with regard to real data collected on a highway. With the traffic simulator, we simulated a road/highway of 50 km with 2 lanes. The desired speed of the vehicles follows a Normal distribution with mean 120 km/h and standard deviation $\sigma = 10$. The distance shown on the x-axis in the figures corresponds to the mean distance between two successive vehicles.

B. Simulation results

d) Results on intensity and capacity: In Figure 3(a), we plotted the mean number of simultaneous transmitters obtained with NS-3 ("NS-3: Simultaneous transmitters with collision" in the figure) and the theoretical limit ("Theoretical number of transmitters - $\frac{\lambda d}{T}$"). $\frac{\lambda d}{T}$ corresponds to formula 7 with $\lambda = \frac{c}{D_{min}}$ the transmitters' intensity, d the highway length (40 km), and T the mean time to send a frame. The traffic increases from one vehicle every 800 meters (1.25 veh/km) to every 100 meters (10 veh/km). We can observe that the number of simultaneous transmitters exceed the theoretical limit. Indeed, we saturated the network to estimate the maximum capacity. The high rate of the CBR sources leads to a significant number of collisions. Since a collision occurs when the CCA rule is not respected (due to the choice of the same back-off by two vehicles for instance), it increases the number of transmitters. But, if we neglect collisions ("NS-3: Simultaneous transmitters without collision" in the figure), the number of transmitters converges to the theoretical value. Unfortunately, it was not possible to consider denser scenarios, as it involved a huge number of vehicles (for a 50 km highway, and 10 veh/km we already have 500 nodes in NS-3). Nevertheless, it appears that this convergence holds even for very low traffic density (10 veh/km corresponding to very good traffic conditions).

The capacity being directly linked to the transmitters' intensity, the capacity obtained by simulations, shown in Figure 3(b), converges to the theoretical one. The capacity fits with the number of transmitters without collisions as we counted only received frames. For this scenario the maximum capacity is then approximately 5200 frames on 40km (we did not count the 5 first and last kilometers of the highway to avoid edge effects) leading to 1300 frames per kilometer and second. From the simulations, we obtained 1113 frames per kilometer and second.

e) Distance between transmitters: Our model gives a theoretical bound on the transmitters' intensity but does not offer analytical results about the distribution of the distance between transmitters. This distribution is important if we want to understand effects of the CSMA/CA on wireless links. Indeed, links properties, in particular interference, strongly depend on transmitters' location. In order to obtain the distribution of the distance between transmitters, we coded a software in C which simulates the model described in section IV. Parameters are the same as in the previous section. In figure 4, we plotted the transmitters inter-distance distribution. For NS-3 simulations, we considered 10 veh/km and three different cases: a case



(a) Simultaneous transmitters

(b) Capacity

Fig. 3. Mean number of simultaneous transmitters and capacity.

Theoretical and NS-3 Parameters	Numerical Values	Theoretical and NS-3 Parameters	Numerical Values
IEEE 802.11std	802.11p - CCH channel	Path-loss function	$l(d) = P_t \cdot min\left(1, \frac{10^{-4.5677}}{d^3}\right)$
CCA mode	CCA mode 1	ED Threshold (θ)	-82 dBm
Emission power P_t	43 dBm	Number of samples per point	100
Length of the packet	1024 bytes	Duration of the simulation	4 sec
DIFS	34 μs	SIFS	16 μs
Road length (d)	50 km		

TABLE I
SIMULATION PARAMETERS.

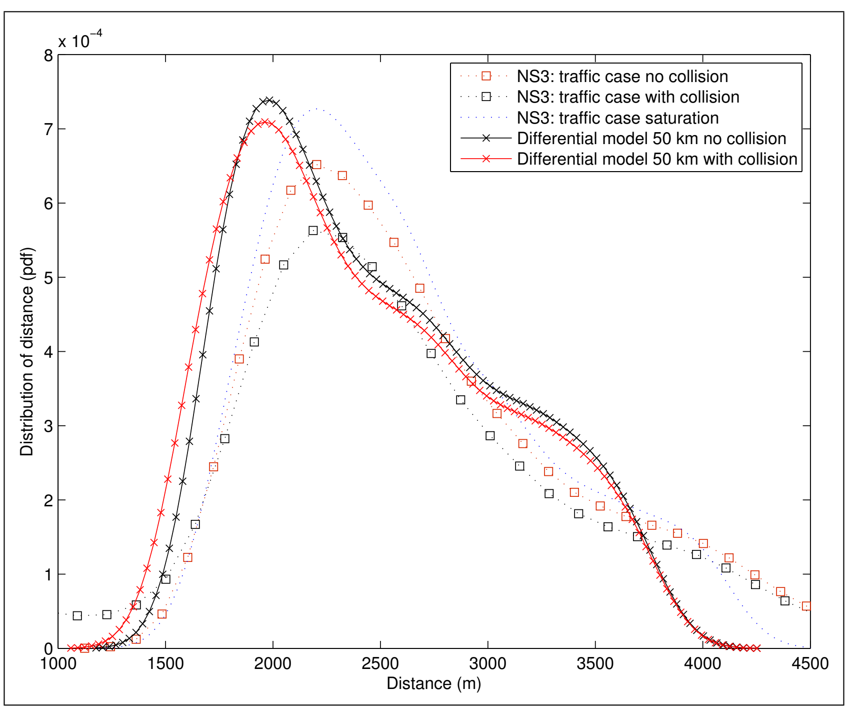


Fig. 4. Distribution of the distances between concurrent transmitters.

with all the transmitters ("NS3: traffic case with collision"), a case where we did not take into account transmitters provoking collisions and a saturated case where we neglected transmitters too far from each others ("NS3: traffic case saturation"). For the latter, we did not consider samples of distance where another transmitter could have been added. The idea was to consider a total spatial saturation of the medium. We observe that all distributions are close to each others. In particular, distributions of the saturated case and the one obtained from the model are very similar: there is only a slight shift between these two curves. It empirically shows that the proposed approach models very well distance between transmitters and corresponds to a case without collisions and where the medium is spatially busy. These conditions are difficult to obtain, even with simulations, it explains the small shift between the two curves.

VI. Conclusion

Capacity of VANET is limited by the spatial reuse of the CSMA/CA mechanism. In this paper, we proposed a simple model to estimate this spatial reuse allowing us to offer an upper bound on the capacity. Realistic simulations that combines the network simulator NS-3 and a vehicles traffic generator have proved that our model offers a very tight bound on the capacity. We have given very simple formula for the estimation of this capacity. This may be useful to design VANET applications, i.e. to set data that will be exchanged between vehicles with regard to the maximum capacity. Also, we compare the transmitters inter-distance distribution obtained with NS-3 and the model. It appears that the distribution obtained from the model is very close to the simulated one. This distribution may help to characterize interference distribution in VANET and to deduce some insight on wireless link properties in VANET. Models and simulations can be improved in two ways. First, the distribution of the distance between transmitters has to be analytically studied or at least extrapolated by known distributions. Also, we are currently working on a model taking into account more realistic assumptions about the radio environment, more precisely taking into account fading and shadowing.

References

[1] Chair: Wayne Fisher editor: Lee Armstrong. Status of project ieee 802.11 task group p. wireless access in vehicular environments (wave). http://grouper.ieee.org/groups/802/11/Reports/tgp_update.htm.

[2] Hannes Hartenstein and Kenneth Kenneth Laberteaux. *VANET Vehicular Applications and Inter-Networking Technologies*. Wiley, 2009.

[3] P. Gupta and P. Kumar. Capacity of wireless networks. *IEEE Transactions on Information Theory*, 46(2):388–404, 2000.

[4] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran. Closing the gap in the capacity of wireless networks via percolation theory. *IEEE Transactions on Information Theory*, 53(3):1009–1018, 2007.

[5] O. Dousse and P. Thiran. Connectivity vs capacity in dense ad hoc networks. In *Conference on Computer Communications (INFOCOM)*, Hong Kong, China, March 2004. IEEE.

[6] V. Mhatre, C. Rosenberg, and R. Mazumdar. On the capacity of ad-hoc networks under random packet losses. *IEEE Transactions on Information Theory*, 55(6):2494–2498, 2009.

[7] Hossein Pishro-Nik, Aura Ganz, and Daiheng Ni. The capacity of vehicular ad hoc networks. In *45th Annual Allerton Conference on Communication, Control and Computing*, Allerton, USA, September 2007.

[8] Mohammad Nekaoui, Ali Eslami, and Hossein Pishro-Nik. Scaling laws for distance limited communications in vehicular ad hoc networks. In *IEEE International Conference on Communications, ICC 2008*, Beijing, China, May 2008.

[9] Lili Du, Satish Ukkusri, Wilfredo F. Yushimito Del Valle, and Shivkumar Kalyanaraman. Optimization models to characterize the broadcast capacity of vehicular ad hoc networks. *Transportation research. Part C, Emerging technologies, Elsevier*, 17(6):571–585, 2009.

[10] A. Busson and G. Chelius. Point processes for interference modeling in csma/ca ad-hoc networks. In *Sixth ACM International Symposium on Performance Evaluation of Wireless Ad Hoc, Sensor, and Ubiquitous Networks (PE-WASUN 2009)*, Tenerife, Spain, October 2009.

[11] P. Hall. *Introduction To the Theory of Coverage Processes*. Wiley, 1988.

[12] Network simulator 3 - ns3. http://www.nsnam.org.

[13] S. Druitt. An introduction to microsimulation. *Traffic engineering and control*, 39(9), 1998.

[14] Kazi I. Ahmed. *Modeling Drivers' Acceleration and Lane Changing Behavior*. PhD thesis, Massachusetts Institute of Technology, 1999.

Appendix

Proof: Proof of Proposition 1. We show that $\frac{m(L)}{L}$ converges to a constant when L tends to infinity. First, we prove that $m(L)$ is a super-additive function, i.e. $m(L) \geq m(s) + m(L-s)$ for all $s \in (0, L)$. If $L < D_{min}$ then $m(L) = m(s) = m(L-s) = 0$ and the assertion is true. To prove the super-additivity for $L > D_{min}$, it suffices to note that, for $s \in [v(L), L - v(L)]$, $m(s)$ and $m(L-s)$ are the mean number of points in the set $[v(s), s - v(s)] \cup [s + v(L-s), L - v(L-s)]$ whereas $m(L)$ is the mean number of points in $[v(L), L - v(L)]$ with $[v(s), s - v(s)] \cup [s + v(L-s), L - v(L-s)] \subset [v(L), L - v(L)]$. For $s \in [0, v(L)]$ (respectively $\in [L - v(L), L]$), $m(s)$ (resp. $m(L-s)$) is nill and the remaining interval $[s + v(L-s), L - v(L-s)]$ (resp. $[v(s), s - v(s)]$) is a subset of $[v(L), L - v(L)]$.

The function $m(L)$ is thus super-additive. According to the Fekete Lemma, $\frac{m(L)}{L}$ converges to a finite or an infinite limit when $L \to +\infty$. To prove that the limit is finite, we show that it exists a positive constant A such that $m(L) \leq AL + 2$. By definition, the minimal distance between two successive points is $\frac{Dmin}{2}$. The mean number of points in $[0, L]$ is thus less than $\frac{D_{min}}{2}L + 2$. The constant 2 is added because $m(L)$ counts the two points at 0 and L. $\frac{m(L)}{L}$ is thus bound by a positive constant ($\frac{D_{min}}{2} + 2$ for instance). Therefore, the limit is finite. ∎