Markov chains, Markov Processes, Queuing Theory and Application to Communication Networks

Anthony Busson,
University Lyon 1 - Lyon - France
anthony.busson@univ-lyon1.fr

November 17, 2016

Contents

1 Markov Chains

1.1 Definitions and properties

Exercise 1. *We consider a flea moving along the points of an axis. The position of the flea on the axis is represented by a positive integer. The flea is initially at the origin* 0*. If the flea is at position* i *(with* $i \neq 0$*) at some stage, it moves with probability* p *(*$0 < p < 1$*) on its right (from* i *to* $i+1$*), and with probability* $1-p$ *on its left. The flea necessarily moves at each stage (it cannot stay at the same location).*

1. *What is the probability that the flea moves to position* 1 *if it is currently at* 0*?*

2. *If after* n *steps, the flea is at position* k*, what is the probability of it being at position* i *at the* $n+1^{th}$ *step?*

3. *What is the probability that the flea is at position* i *at the* $n+1^{th}$ *step without having knowledge of the past?*

Answers

1. *The flea has only one possibility. It can only move to position* 1*. Therefore, the probability to move to* 1 *is* 1 *and hence the probability to move to any other position is* 0*.*

2. *The probability to move from position* k *to* $k+1$ *is* p *and the probability to move from* k *to* $k-1$ *is* $1-p$ *(that's given by the subject of the exercise). For any other positions, the probability is nill.*

3. *If we know only the initial location of the flea (it is* 0 *for this exercise), we have to look at all the possible trajectories from* 0 *to* i *in* $n+1$ *steps. The Markov chain offers a Mathematical (and practical) framework to do that.*

Definition 1. ***Markov Chain.*** *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of random variables taking values in a countable space* E *(in our case we will take* $\mathbb{N}$ *or a subset of* $\mathbb{N}$*).* $(X_n)_{n\in\mathbb{N}}$ *is a* ***Markov chain*** *if and only if*

$$\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, .., X_0 = i_0) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1})$$

Thus for a Markov chain, the state of the chain at a given time contains all the information about the past evolution which is of use in predicting its future behavior. We can also say that given the present state, the past and the future are independent.

Definition 2. ***Homogeneity.*** *A Markov chain is* ***homogeneous*** *if and only if* $\mathbb{P}(X_n = j | X_{n-1} = i)$ *does not depend on* n*.*

Definition 3. ***Irreducibility.*** *A Markov chain is* ***irreducible*** *if and only if every state can be reached from every other state.*

Definition 4. *Transition probabilities. For a homogeneous Markov chain,*

$$p_{j,i} = \mathbb{P}(X_n = i | X_{n-1} = j)$$

is called the ***transition probability*** *from state j to state i.*

Definition 5. ***Period.*** *A state i is* ***periodic*** *if there exists an integer $\delta > 1$ such that $\mathbb{P}(X_{n+k} = i | X_n = i) = 0$ unless k is divisible by δ ; otherwise the state is* ***aperiodic****. If all the states of a Markov chain are periodic (respectively aperiodic), then the chain is said to be periodic (respectevily aperiodic).*

Exercise 2. *We consider a player playing "roulette" in a casino. Initially the player has N dollars. For each spin, he bets 1 dollar on red. Let X_n be the process describing the amount of money that the player has after n bets.*

1. *Is X_n a Markov chain?*
2. *In case of a Markov chain, what are the transition probabilities? Are the states periodic or aperiodic? Is it irreducible?*
3. *What is the distribution of X_n with regard to X_{n-1}, and X_n with regard to X_{n-2} ?*

Answers

1. *X_n is a Markov chain. Given the value of X_{n-1}, we can express the distribution of X_n whatever the value of X_k for $k < n - 1$.*
2. *The transition probabilities are*

$$p_{i,i+1} = \frac{\textit{Number of red compartments}}{\textit{Total number of compartments}}$$

$$p_{i,i-1} = \frac{\textit{Number of black and green compartments}}{\textit{Total number of compartments}}$$

The number of compartments is 37 in France and 38 in USA. Both have 18 red and black compartments but in USA there are two zeros (0 and 00). So, in France, we get $p_{i,i+1} = \frac{18}{37}$ and $p_{i,i-1} = \frac{19}{37}$. For the USA, we get $p_{i,i+1} = \frac{18}{38}$ and $p_{i,i-1} = \frac{20}{38}$.
The period can be seen as the minimum number of steps required to return to the same state. For the above case, we can come back at state j only after an even number of steps (except for the state 0). The period is then 2 for all the states (except state 0). The chain is not irreducible (irreducible means that for all (i, j), there is at least

one trajectory from i to j with positive probability). Indeed, once X_n has reached 0, the player hasn't got anymore money. The state 0 is therefore an absorbant state since the chain stays in this state for the rest of the time.

3. *The distribution of X_n with regard to X_{n-1} is given by the transition probabilities. The distribution of X_n w.r.t. X_{n-2} is obtained by conditionning the possible values of X_{n-1} (if we look at the probability that $X_n = i$, the only possible values of X_{n-1} for which the probability is not 0 is $X_{n-1} = i-1$ and $X_{n-1} = i+1$):*

$$\begin{aligned}
&\mathbb{P}\left(X_n = i | X_{n-2} = j\right) \\
&= \mathbb{P}\left(X_n = i, X_{n-1} = i-1 | X_{n-2} = j\right) + \mathbb{P}\left(X_n = i, X_{n-1} = i+1 | X_{n-2} = j\right) \\
&= \mathbb{P}\left(X_n = i | X_{n-1} = i-1, X_{n-2} = j\right) \mathbb{P}\left(X_{n-1} = i-1 | X_{n-2} = j\right) \\
&\quad + \mathbb{P}\left(X_n = i | X_{n-1} = i+1, X_{n-2} = j\right) \mathbb{P}\left(X_{n-1} = i+1 | X_{n-2} = j\right) \text{ using Baye's formula} \\
&= \mathbb{P}\left(X_n = i | X_{n-1} = i-1\right) \mathbb{P}\left(X_{n-1} = i-1 | X_{n-2} = j\right) \\
&\quad + \mathbb{P}\left(X_n = i | X_{n-1} = i+1\right) \mathbb{P}\left(X_{n-1} = i+1 | X_{n-2} = j\right) \text{ using Markov property} \\
&= p_{i-1,i} p_{j,i-1} + p_{i+1,i} p_{j,i+1}
\end{aligned}$$

The only values of j for which the equality above will not be nill are $j = i-2, i, i+2$. Let note $p_{i,i-1} = p$ and $p_{i,i+1} = 1-p$ (it does not depend on i), we get

$$\begin{aligned}
\mathbb{P}\left(X_n = i | X_{n-2} = i-2\right) &= p^2 \\
\mathbb{P}\left(X_n = i | X_{n-2} = i\right) &= 2p(1-p) \\
\mathbb{P}\left(X_n = i | X_{n-2} = i+2\right) &= (1-p)^2
\end{aligned}$$

Exercise 3. *Give an example of a periodic Markov chain.*

Answer *We have seen that the Markov chain of exercise 2 has all its states periodic with period 2 except 0. To have all the states of the same period, we just have to change the transition probability of state 0 in such a way that its period is 2. Let $p_{0,1} = 1$ (rather than $p_{0,0} = 1$) and we obtain a Markov chain with period 2. Note that if the chain is irreducible, all the states have the same period. When we changed the transition probability of state 0, the chain became irreducible.*

1.2 Distribution of a Markov chain.

Definition 6. *We define the distribution vector V_n of X_n as*

$$V_n = (\mathbb{P}(X_n = 0), \mathbb{P}(X_n = 1), \mathbb{P}(X_n = 2), ..)$$

Exercise 4. *Let V_n be the distribution vector of a Markov chain X_n.*

1. *Express vector V_n as function of V_{n-1} and the transition matrix P.*

2. *Express V_n as function of V_0 and P.*

Answers

1. *We consider a Markov chain X_n which takes its values from $\{0, 1, 2, .., N\}$ (the method would be the same for any countable space) with transition matrix P. We are going to express $V_n(i)$ with regard to V_{n-1} and P. Conditionning by the values of X_{n-1}, we obtain:*

$$\begin{aligned} V_n(i) &= \mathbb{P}(X_n = i) \\ &= \sum_{j=0}^{N} \mathbb{P}(X_n = i, X_{n-1} = j) \\ &= \sum_{j=0}^{N} \mathbb{P}(X_n = i | X_{n-1} = j) \mathbb{P}(X_{n-1} = j) \quad \textit{Baye's formulae} \\ &= \sum_{j=0}^{N} p_{j,i} V_{n-1}(j) \end{aligned}$$

The above equation can be easily express in matrix form:

$$V_n = V_{n-1} P$$

2. *A simple recursive process gives the result:*

$$\begin{aligned} V_n &= V_{n-1} P \\ &= V_{n-2} P^2 \\ &= V_0 P^n \end{aligned}$$

So a simple way to compute the probability that $\{X_n = i\}$ is to compute $\sum_{j \in E} V_0(j) p_{j,i}^{(n)}$. More generally, we get (if we stop the recurrence after k steps):

$$V_n(i) = \sum_{j \in E} V_{n-k}(j) p_{j,i}^{(n-k)} \textit{(The Chapman-Kolmogorov equation)}$$

It is important to note that $p_{j,i}^{(k)}$ is the element (i, j) of the matrix P^k and not the element $p_{j,i}$ to the power k of the matrix P.

Exercise 5. *We consider a Markov chain that take its value from the set $\{0, 1\}$. The transition matrix is*

$$\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

What is the limit of V_n when n tends to infinity? Compute V_n with $p = q = \frac{1}{2}$.

Answers *From exercise 4, we know that $V_n = V_0 P^n$. One way to compute V_n is to "diagonalize" the matrix P. In other words, we break up P as $P = QDQ^{-1}$ where D is a diagonal matrix and Q is invertible. Hence, we can write*

$$V_n = V_0 Q D^n Q^{-1}$$

The eigenvalues of P are the roots of the equation $det(P - \lambda I) = 0$, where $det()$ is the determinant, I is the identity matrix with same dimensions as that of P and λ is a scalar. We obtain for our example:

$$det(P - \lambda I) = \lambda^2 - \lambda(p+q) - (1-p-q)$$

The two roots of this equation are $\lambda_1 = 1$ and $\lambda_2 = -(1-q-p)$. The matrix D is therefore

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -(1-q-p) \end{pmatrix}$$

The eigenvectors can be found by solving the equation $(P - \lambda_1 I)X = 0$ (resp. λ_2) with $x = (x_1, x_2)$. For λ_1, we get

$$(p-1)x_1 + (1-p)x_2 = 0$$
$$(1-q)x_1 + (q-1)x_2 = 0$$

The two equations are redundant, it suffices to find a vector (x_1, x_2) which verifies one of the two equations. We choose $x_1 = 1$ and $x_2 = 1$. The first eigenvector is therefore $E_1 = (1,1)^T$. Similar computations lead to $E_2 = (1, -\frac{1-q}{1-p})^T$. The matrix Q and Q^{-1} are then

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{1-q}{1-p} \end{pmatrix}$$

and

$$Q^{-1} = \begin{pmatrix} \frac{1-q}{1-p+1-q} & \frac{1-p}{1-p+1-q} \\ \frac{1-p}{1-p+1-q} & -\frac{1-p}{1-p+1-q} \end{pmatrix}$$

From these expressions, it is very easy to compute V_n for all n. If $p = q = \frac{1}{2}$, we get

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Since for this particular case, $D^n = D$, we obtain

$$V_n = V_0 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and $V_n = (\frac{1}{2}, \frac{1}{2})$ whatever the values of V_0. We have seen through this simple example, that the diagonalization of transition matrix is not an easy task. When the space E has a much greater number of elements (note that it can also be infinite), the diagonalization is even impossible. Moreover, in practical cases, we are not interested in the distribution of V_n for small n but rather for large n or for the asymptotic behavior of the Markov chain. We will see that under certain assumptions, the distribution of V_n reaches an equilibrium state which is clearly easier to compute.

Definition 7. *If X_n has the same distribution vector as X_k for all $(k, n) \in \mathbb{N}^2$, then the Markov chain is stationary.*

Exercise 6. *Let $(X_n)_{n \in \mathbb{N}}$ be a stationary Markov chain. Find an equation that the distribution vector should verify.*

Theorem 1. *Let $(X_n)_{n \in \mathbb{N}}$ be a Markov chain. We assume that X_n is irreducible, aperiodic and homogeneous. The Markov chain may possess an* ***equilibrium distribution*** *also called* ***stationary distribution*** *, that is, a distribution vector π (with $\pi = (\pi_0, \pi_1, ...)$) that satisfies*

$$\pi P = \pi \tag{1}$$

and

$$\sum_{k \in E} \pi_k = 1 \tag{2}$$

If we can find a vector π satisfying equations (1) and (2), then this distribution is unique and

$$\lim_{n \to +\infty} \mathbb{P}\left(X_n = i | X_0 = j\right) = \pi_i$$

so that π is the limiting distribution of the Markov chain.

Theorem 2. *Ergodicity. Let $(X_n)_{n \in \mathbb{N}}$ be a Markov chain. We assume that X_n is irreducible, aperiodic and homogeneous. If a Markov chain possesses* *an equilibrium distribution, then the proportion of time the Markov chain spends in state i during the period $[0, n]$ converges to π_i as $n \to +\infty$:*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbb{1}_{X_k=i} = \pi_i$$

In this case, we say that the Markov chain is ergodic.

Exercise 7. *In this exercise, we give two examples of computation and interpretation of the distribution obtained with equations 1 and 2 when the assumptions of theorem 1 does not hold.*

1. *Resume exercise 2 on roulette. Does the Markov chain possess an equilibrium distribution? Compute it.*
2. *Resume exercise 1 on the flea. Does the Markov chain possess an equilibrium distribution? Compute it.*

Remark 1. ***Equilibrium distribution and periodicity.*** *If the Markov chain is periodic, then there is no limit to the distribution vector. The solution of equations (1) and (2) when it exists, is interpreted as the proportion of time that the chain spends in the different states.*

Remark 2. ***Transient chain.*** *If there is no solution to equations (1) and (2) for an aperiodic, homogeneous, irreducible Markov chain $(X_n)_{n\in\mathbb{N}}$ then there is no equilibrium distribution and*

$$\lim_{n\to+\infty} \mathbb{P}(X_n = i) = 0$$

Remark 3. *The state of a Markov chain may be classified as transient or recurrent.*

1. *A state i is said to be transient if, given that we start in state i, there is a non-zero probability that we will never return back to i. Formally, let the random variable T_i be the next return time to state i ($T_i = \min\{n : X_n = i | X_0 = i\}$), then state i is transient if and only if there exists a finite T_i such that*

$$\mathbb{P}(T_i < +\infty) < 1$$

2. *A state i will be recurrent if it is not transient ($\mathbb{P}(T_i < +\infty) = 1$).*

In case of an irreducible Markov chain, all the states are either transient or recurrent. If the Markov chain possesses an equilibrium distribution, then the Markov chain is recurrent (all the states are recurrent) and $\pi_i = \frac{1}{\mathbb{E}[T_i]}$.

Exercise 8. *Resume exercise 1 on the flea. We assume now, that there is a probability p that the flea goes to the left, and a probability q that the flea stays at the current state (with $p + q < 1$).*

1. *Express the transition matrix.*
2. *Is the Markov chain still periodic?*
3. *What is the condition of existence of a solution to equations (1) and (2)? Compute it.*

Answers

The transition matrix is:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 1-q-p & q & p & 0 & 0 & \ldots \\ 0 & 1-q-p & q & p & 0 & \ldots \\ 0 & 0 & 1-q-p & q & p & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Equation $\pi P = \pi$ leads to the following set of equations:

$$\begin{aligned} \pi_1(1-q-p) &= \pi_0 && (3) \\ \pi_0 + q\pi_1 + (1-q-p)\pi_2 &= \pi_1 && (4) \\ p\pi_1 + q\pi_2 + (1-q-p)\pi_3 &= \pi_2 \\ &\ldots \\ p\pi_{k-1} + q\pi_k + (1-q-p)\pi_{k+1} &= \pi_k && (5) \end{aligned}$$

From equations 3 and 4, we obtain

$$\pi_1 = \frac{\pi_0}{1-q-p} \text{ and } \pi_2 = \frac{p}{(1-q-p)^2}\pi_0$$

We assume that π_k can be written in the form $\pi_k = \frac{p^{k-1}}{(1-q-p)^k}\pi_0$. To prove this, we just have to verify that this form is a solution of equation 5 (it is of course true). Now, the distribution has to verify equation 2:

$$\begin{aligned} \sum_{i=0}^{+\infty} \pi_i &= \left[1 + \sum_{i=1}^{+\infty} \frac{p^{i-1}}{(1-q-p)^i}\right]\pi_0 \\ &= \left[1 + \frac{1}{(1-q-p)} \sum_{i=0}^{+\infty} \left(\frac{p}{(1-q-p)}\right)^i\right]\pi_0 \end{aligned}$$

The sum in the above equation is finite if and only if $p < 1 - q - p$. In this case, we get (from $\sum \pi_1 = 1$):

$$\pi_0 = \left[1 + \frac{1}{1-q-2p}\right]^{-1}$$

and

$$\pi_k = \pi_0 \frac{p^{k-1}}{(1-q-p)^k}$$

If $p > 1-q-p$, the sum is infinite and $\pi_0 = 0$, thus $\pi_i = 0$ for all i. The process is then transient. The flea tends to move away indefinitely from 0.

Exercise 9. *Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with $E = 0, 1, 2, 3, 4$ and with the following transition matrix:*

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

1. *Solve equations (1) and (2) for this Markov chain.*

2. *What is the condition for the existence of a solution? Why?*

Answers *The equation $\pi P = \pi$ leads to the following set of equations:*

$$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_0$$

$$\frac{1}{2}\pi_0 + \frac{1}{3}\pi_2 = \pi_1$$

$$\frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 = \pi_2 \quad (6)$$

$$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_3 \quad (7)$$

$$\frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_4 \quad (8)$$

Equation 8 leads to $\pi_3 = \pi_4$. Substituting $\pi_3 = \pi_4$ in equation 7 leads to $\pi_1 = \pi_2 = 0$ (since π_1 and π_2 cannot be negative). Putting $\pi_1 = \pi_2 = 0$ in equation 6 leads to $\pi_0 = 0$. Finally, we get $\pi_0 = \pi_1 = \pi_2 = 0$ and $\pi_3 = \pi_4 = \frac{1}{2}$. This is easily interpretable. If the chain goes in the state 3 or 4, it stays indefinitly among these two states. Indeed, the probability to go to other states is nill. So, if the chain starts from a state among $\{0, 1, 2\}$, it will stay in it for a finite number of steps, then (with probability 1) it will move to the set $\{3, 4\}$ and stay there for the rest of the time. The states $\{0, 1, 2\}$ are transient and the other two states are recurrent. It is due to the fact that the chain is not irreducible. When we deal with a reducible chain, we generally consider only a subset of states (which are irreducible) and we compute the stationnary distribution for this subset. In this exercise, the irreducible subset of states is $\{3, 4\}$.

1.3 A first queue

Exercise 10. *Let us consider the following discrete time queue. We consider a transmission buffer where packets are stored before emission on a link. The time is slotted. At each slot, the system tries to send a packet. Since the link is considered unreliable, there is a probability p that a packet is lost. In this case, the packet stays in the buffer and a new attempt is made at the next slot. The packet arrives in the buffer with a rate a. In other words, a is the probability that a new packet arrives at a slot. Packet arrivals and losses are supposed independent at each slot.*

Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of random variables representing the number of packets in the buffer at slot n. We assume that initially there are no packets in the buffer ($X_0 = 0$).

1. *Is X_n a Markov chain?*
2. *Compute the transition Matrix.*
3. *What is the existence condition of an equilibrium distribution?*
4. *Compute the equilibrium distribution when it exists.*

Solution. *Exercise 10*

1. *The discrete process X_n is a Markov chain by definition.*
2. *The transition Matrix is as follows:($p_{0,0} = 1-a, p_{0,1} = a$ and for $i > 0$ $p_{i,i-1} = (1-p)(1-a)$, i.e. no packet has came and a packet has been transmitted; $p_{i,i} = (1-a)p + a(1-p)$, i.e. no packet has came and no packet has been transmitted (it has been lost and as a consequence it stays in the queue); $p_{i,i+1} = ap$, a packet has came but no packet has been transmitted):*

$$\begin{pmatrix} 1-a & a & 0 & ... & & & \\ (1-a)(1-p) & (1-a)p+a(1-p) & ap & 0 & ... & & \\ 0 & (1-a)(1-p) & (1-a)p+a(1-p) & ap & 0 & ... \\ & & ... & ... & ... & \end{pmatrix}$$

3 and 4. *We have to solve equations (1) and (2). Since the chain is aperiodic, homogeneous and irreducible, it suffices that there is a solution to prove the existence and uniqueness of an equilibrium/limit distribution. The equations $\pi P = \pi$ leads to the following equations system:*

$$\begin{aligned}
\pi_0 &= (1-a)\pi_0 + (1-a)(1-p)\pi_1 && (9)\\
\pi_1 &= a\pi_0 + [(1-a)p + a(1-p)]\pi_1 + (1-p)(1-a)\pi_2 && (10)\\
... &= ... \\
\pi_k &= ap\pi_{k-1} + [(1-a)p + a(1-p)]\pi_k + (1-p)(1-a)\pi_{k+1} && (11)
\end{aligned}$$

Equation 9 (resp. 10) leads to the expression of $\pi 1$ (resp. π_2) as a function of π_0:

$$\pi_1 = \frac{a}{(1-a)(1-p)}\pi_0$$

$$\pi_2 = \frac{a^2 p}{(1-a)^2(1-p)^2}\pi_0$$

So, we assume that π_k has the following form:

$$\pi_k = \frac{a^k p^{k-1}}{(1-a)^k(1-p)^k}\pi_0$$

We prove that by substituting this expression in the left-hand side of equation (11):

$$\begin{aligned}
(11) \quad &= ap\frac{a^{k-1}p^{k-2}}{(1-p)^{k-1}(1-a)^{k-1}}\pi_0 + [(1-a)p + a(1-p)]\frac{a^k p^{k-1}}{(1-p)^k(1-a)^k}\pi_0 \\
&\quad +(1-p)(1-a)\frac{a^{k+1}p^k}{(1-p)^{k+1}(1-a)^{k+1}}\pi_0 \\
&= \pi_0\frac{a^k p^{k-1}}{(1-p)^k(1-a)^k}\left[(1-a)(1-p) + (1-a)p + a(1-p) + ap\right] \\
&= \pi_0\frac{a^k p^{k-1}}{(1-p)^k(1-a)^k}\left[(1-a) - (1-a)p + (1-a)p + a(1-p) + ap\right] \\
&= \pi_0\frac{a^k p^{k-1}}{(1-p)^k(1-a)^k}\left[1\right]
\end{aligned}$$

The sum must be one.

$$\begin{aligned}
\sum_{k=0}^{+\infty}\pi_k &= \pi_0\left[1 + \sum_{k=1}^{+\infty}\frac{1}{p}\left(\frac{ap}{(1-a)(1-p)}\right)^k\right] \\
&= \pi_0\left[1 + \frac{1}{p}\left(\frac{ap}{(1-a)(1-p)}\right)\sum_{k=0}^{+\infty}\left(\frac{ap}{(1-a)(1-p)}\right)^k\right] \\
&= \pi_0\left[1 + \frac{1}{p}\left(\frac{ap}{(1-a)(1-p)}\right)\frac{1}{1-\frac{ap}{(1-a)(1-p)}}\right]
\end{aligned}$$

The last equality is true only if the term to the power k is strictly less than 1. Otherwise the sum does not converge. The condition for the existence of an equilibrium distribution is then $ap < (1-a)(1-p)$ leading to $1-p > a$. So, the probability of arrival MUST BE less than the probability of transmission success. It is obvious that otherwise, the buffer will fill infinitely. The last equation gives the expression for π_0 and the final expression of π_k depends only on p and a.

2 Markov Processes

In this Section, we introduce the definition and main results for Markov processes. A Markov process is a random process $(X_t)_{t\in\phi}$ indexed by a continuous space (it is indexed by a discrete space for the Markov chains) and taking values in a countable space E. In our case, it will be indexed by $I\!R^+$ ($\phi = I\!R^+$). In this course, we are only interested in non explosive Markov processes (also called regular processes), i.e. processes which are capable of passing through an infinite number of states in a finite time. In the definition below, we give the property that a random process must verify to be a Markov process.

Definition 8. *The random process or stochastic process $(X_t)_{t\in I\!R^+}$ is a* ***Markov process*** *if and only if, for all $(t_1, t_2, ...t_n) \in (I\!R^+)^n$ such that $t_1 < t_2 < t_3 < ... < t_{n-1} < t_n$ and for $(i_1, ..., i_n) \in E^n$,*

$$\mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}, \quad X_{t_{n-2}} = i_{n-2}, ..., X_{t_1} = i_1) = \mathbb{P}\left(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}\right)$$

A Markov process is ***time homogeneous*** *if $\mathbb{P}\left(X_{t+h} = j | X_t = i\right)$ does not depend on t. In the following, we will consider only homogeneous Markov process.*

The interpretation of the definition is the same as in the discrete case (Markov chain) ; Conditionally to the present the past and the future are independent. From the Definition 8 of a Markov process, we obtain in Sections 2.2 and 2.3 the main results on the Markov processes. But, we begin by giving some properties of the exponential distribution which play an important part in Markov process.

2.1 Properties of the exponential distribution.

Definition 9. *The* ***probability density function*** *(pdf) of an exponential distribution with parameter μ ($\mu \in I\!R^+$) is*

$$f(x) = \mu e^{-\mu x} \mathbb{1}_{x\geq 0}$$

The ***cumulative distribution function*** *(denoted cdf, defined as $F(u) = \mathbb{P}(T \leq u)$) of an exponential distribution with parameter μ is*

$$F(u) = \left(1 - e^{-\mu u}\right) \mathbb{1}_{u\geq 0}$$

Proposition 1. ***Memorylessness.*** *An important property of the exponential distribution is that it is memoryless. This means that if a random variable T is exponentially distributed, its conditional probability obeys*

$$\mathbb{P}(T > t + s | T > s) = \mathbb{P}(T > t)$$

Intuitively, it means that the conditional probability to wait more than t after waiting s, is no different from the initial probability to wait more than t.

Exercise 11. *Prove Proposition 1.*

Proof. By definition of the conditional probability, we get

$$\begin{aligned}
\mathbb{P}(T > t + s | T > s) &= \frac{\mathbb{P}(T > t + s, T > s)}{\mathbb{P}(T > s)} \\
&= \frac{\mathbb{P}(T > t + s)}{\mathbb{P}(T > s)} \\
&= \frac{e^{-\mu(t+s)}}{e^{-\mu s}} \\
&= e^{-\mu t}
\end{aligned}$$

Thus, $\mathbb{P}(T \leq t + s | T > s) = 1 - e^{-\mu t}$ which is the cdf of an exponential distribution. □

Proposition 2. *If a random variable T is memoryless, i.e. if it verifies the following property*

$$\mathbb{P}(T > t + s | T > s) = \mathbb{P}(T > t)$$

then it follows an exponential distribution.

Proof. Let T be a random variable satisfying the following property:

$$\mathbb{P}(T > t + s | T > s) = \mathbb{P}(T > t)$$

$$\begin{aligned}
\mathbb{P}(T > t + s) &= \mathbb{P}(T > t + s, T > s) \\
&= \mathbb{P}(T > t + s | T > s)\, \mathbb{P}(T > s) \\
&= \mathbb{P}(T > t)\, \mathbb{P}(T > s) \text{ Memorylessness of } T
\end{aligned}$$

The last equality is only verified by exponential distribution. Indeed, $\forall n \in \mathbb{N}^+$, we get

$$\mathbb{P}(T > t) = \mathbb{P}\left(T > \frac{t}{n}\right)^n$$

There is only one function satisfying the last equality for all n, i.e. the power function. Therefore, there exists a constant $\mu > 0$ such that $\mathbb{P}(T > t) = e^{-\mu t}$ for all $t > 0$. The random variable T definitely follows an exponential distribution. □

Proposition 3. ***Minimum of exponential r.v.*** *Let $X_1, ..., X_n$ be independent exponentially distributed random variables with parameters $\mu_1, ..., \mu_n$. Then* $\min(X_1, ..., X_n)$ *is also exponentially distributed with parameter $\mu_1 + \mu_2 + ... + \mu_n$.*

The probability that X_i is the minimum is $\frac{\mu_i}{\mu_1+\mu_2+...+\mu_n}$.

Exercise 12. *Prove Proposition 3.*

Proof. The proof is obtained by recurrence. Let X_1 and X_2 be independent exponentially distributed random variables with parameters μ_1, μ_2.

$$\begin{aligned}
\mathbb{P}\left(min(X_1, X_2) > u\right) &= \mathbb{P}\left(X_1 > u, X_2 > u\right) \\
&= \mathbb{P}\left(X_1 > u\right) \mathbb{P}\left(X_2 > u\right) \\
&= e^{-\mu_1 u} e^{-\mu_2 u} \\
&= e^{-(\mu_1+\mu_2)u}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(argmin(X_1, X_2) = X_1\right) &= \mathbb{P}\left(X_2 > X_1\right) \\
&= \int_0^{+\infty} \mathbb{P}\left(X_2 > u\right) f_{X_1}(u) du \\
&= \int_0^{+\infty} e^{-\mu_2 u} \mu_1 e^{-\mu_1 u} du \\
&= \frac{\mu_1}{\mu_1 + \mu 2}
\end{aligned}$$

$\min(X_1, X_2, X_3) = \min(\min(X_1, X_2), X_3)$ is the min of two independent exponential distribution with parameters $\mu_1 + \mu_2$ and μ_3. Thus it is an exponential distribution with parameter $\mu_1 + \mu_2 + \mu_3$, and so on.

□

2.2 Distribution of a Markov process.

From the definition of a homogeneous Markov process we can deduce that the time that the Markov process spent in a given state is exponentially distributed. Indeed, by definition of a Markov process the process is memory less (the proof is quite similar to the proof of Proposition 2).

Proof. Let Y_i be the time spent in state i. We show that Y_i is memoryless and thus exponentially distributed according to Proposition 2.

$$
\begin{aligned}
\mathbb{P}(Y_i > u+v | Y_i > v) &= \mathbb{P}(X_t = i, \forall t \in [0, u+v] | X_t = i, \forall t \in [0, v]) \\
&= \mathbb{P}(X_t = i, \forall t \in [v, u+v] | X_t = i, \forall t \in [0, v]) \\
&= \mathbb{P}(X_t = i, \forall t \in [v, u+v] | X_v = i) \text{ (Markov property)} \\
&= \mathbb{P}(X_t = i, \forall t \in [0, u] | X_0 = i) \text{ (Homogeneity)} \\
&= \mathbb{P}(Y_i > u)
\end{aligned}
$$

□

Given the current state of the Markov process, the future and past are independent, the choice of the next state is thus probabilistic. The probabilities of change of states can thus be characterized by transition probabilities (a transition matrix). Since the Markov process is supposed homogeneous, these transition probabilities do not depend on time but only on the states. So, we give a new definition of a Markov process which is of course equivalent to the previous one.

Definition 10. *A homogeneous Markov process is a random process indexed by $\mathbb{R}^+$ taking values in a countable space E such that:*

- *the time spent in state i is exponentially distributed with parameter μ_i,*
- *transition from a state i to another state is probabilistic (and independent of the previous steps/hops). The transition probability from state i to state j is $p_{i,j}$.*

Exercise 13. *Let us consider the following system. It is a queue with one server. The service duration of a customer is exponentially distributed with parameter μ and independent between customers. The interarrival times between customers are exponentially distributed with parameters λ. An interarrival is supposed independent of the others interarrivals and service durations.*

1. *Prove that the time spent in a given state is exponentially distributed. Compute the parameter of this distribution.*
2. *Compute the transition probability.*

Answers

1. *Suppose that at time T the Markov process has changed its state and is now equal to i ($i > 0$), so we get $X_T = i$. Since two customers cannot leave or arrive at the same time, there is almost surely one customer who has left (or who has arrived) at T. Suppose that a customer has arrived at T, then the time till the next arrival follows an exponential distribution with parameter λ. Let U denote the random*

variable associated with this distribution. Due to the memorylessness of the exponential distribution, the next departure is still an exponential distribution with parameter μ. Let V be this random variable. The next event will arrive at $min(U,V)$. Since U and V are independent, $min(U,V)$ is also an exponential random variable with parameter $\mu+\lambda$. The same result holds if a customer leaves the system at T.

2. *Suppose that there are i customers in the system. The next event will correspond to a departure or an arrival. It will be an arrival if $min(U,V) = U$ and it will be a departure if $min(U,V) = V$. From Proposition 3, we deduce that the probability to move from i to $i+1$ ($i > 0$) is then $\mathbb{P}\left(min(U,V) = U\right) = \frac{\lambda}{\lambda+\mu}$ and the probability to move from i to $i-1$ is $\mathbb{P}\left(min(U,V) = V\right) = \frac{\mu}{\lambda+\mu}$. The probability to go from 0 to 1 is 1.*

Let V_t be the distribution vector of the Markov process. V_t is defined as

$$V_t = \left(\mathbb{P}\left(X_t = 0\right), \mathbb{P}\left(X_t = 1\right), \mathbb{P}\left(X_t = 2\right), ...\right)$$

In order to find an expression for $V(t)$, we try to compute its derivative. Let h be a small value. The probability to have more than one change between t and $t+h$ belongs to $o(h)$ (set of functions f such that $\frac{f(h)}{h} \to 0$ when $h \to 0$). For instance, the probability that there are two hops between t and $t+h$ from state i to j and from j to k is (where Y_i is time spent in state i)

$$\begin{aligned}
\mathbb{P}\left(Y_i + Y_j \le h\right) &= \int_0^h \mathbb{P}\left(Y_i \le u\right) \mu_j e^{-\mu_j u} du \\
&= \int_0^h \left(1 - e^{-\mu_i(h-u)}\right) \mu_j e^{-\mu_j u} du \\
&= 1 + \frac{\mu_i}{\mu_j - \mu_i} e^{-\mu_j h} - \frac{\mu_j}{\mu_j - \mu_i} e^{-\mu_i h} \\
&= 1 + \frac{\mu_i}{\mu_j - \mu_i}\left(1 - \mu_j h + \frac{(\mu_j h)^2}{2} + o(h)\right) \\
&\quad - \frac{\mu_j}{\mu_j - \mu_i}\left(1 - \mu_i h + \frac{(\mu_i h)^2}{2} + o(h)\right) \\
&= \frac{\mu_i \mu_j}{2} h^2 + o(h) \\
&= o(h)
\end{aligned}$$

The probability that the process jumps from state i to state j between t and $t+h$ can be written as the sum of two quantities. The first one is the probability that the process jumps from i to j in one hop; the second quantity is the probability that the process jumps form i to j between t and $t+h$ with more than one hops. For this last quantity we have seen that it belongs to $o(h)$. From Definition 10, the first quantity can be written as the product that there is one hop betweeen t and $t+h$ (that's the probability that the exponential variable describing the time spent in state i is less than h) multiplied by its transition probability (probability to go from i to j).

$$\begin{aligned}\mathbb{P}\left(X_{t+h}=j|X_t=i\right) &= p_{i,j}\mathbb{P}\left(Y_i<h\right)+o(h)\\ &= p_{i,j}(1-e^{-\mu_i h})+o(h)\\ &= p_{i,j}\left(1-\left(1-\mu_i h+o\left(h\right)\right)\right)+o(h)\\ &= p_{i,j}\mu_i h+o(h)\end{aligned}$$

In the same way, the probability that the process is in the same state i at time t and $t+h$ is

$$\begin{aligned}\mathbb{P}\left(X_{t+h}=i|X_t=i\right) &= \mathbb{P}\left(Y_i>h\right)+o(h)\\ &= e^{-\mu_i h}+o(h)\\ &= 1-\mu_i h+o(h)\end{aligned}$$

Now, we compute the derivative of a term of the distribution vector, by definition of a derivative, we get

$$\frac{dV_t(j)}{dt}=\lim_{h\to 0}\frac{V_{t+h}(j)-V_t(j)}{h}=\lim_{h\to 0}\frac{\mathbb{P}\left(X_{t+h}=j\right)-\mathbb{P}\left(X_t=j\right)}{h}$$

Therefore,

$$\begin{aligned}V_{t+h}(j) &= \mathbb{P}\left(X_{t+h}=j\right)\\ &= \sum_{i\in E}\mathbb{P}\left(X_{t+h}=j|X_t=i\right)\mathbb{P}\left(X_t=i\right)\\ &= \sum_{i\in E}\mathbb{P}\left(X_{t+h}=j|X_t=i\right)V_t(i)\\ &= \mathbb{P}\left(X_{t+h}=j|X_t=j\right)V_t(j)+\sum_{i\in E,i\neq j}\mathbb{P}\left(X_{t+h}=j|X_t=i\right)V_t(i)\\ &= \left(1-\mu_j h+o(h)\right)V_t(j)+\sum_{i\in E,i\neq j}\left(p_{i,j}\mu_i h+o(h)\right)V_t(i)\end{aligned}$$

$$\begin{aligned}V_{t+h}(j)-V_t(j) &= \left(-\mu_i h+o(h)\right)V_t(j)+\sum_{i\in E,i\neq j}\left(p_{i,j}\mu_i h+o(h)\right)V_t(i)\\ &= -\mu_j hV_t(j)+\sum_{i\in E,i\neq j}p_{i,j}\mu_i hV_t(i)+o(h)\end{aligned}$$

and

$$\lim_{h\to 0}\frac{V_{t+h}(j)-V_t(j)}{h} \quad = \quad -\mu_j V_t(j) + \sum_{i\in E, i\neq j} p_{i,j}\mu_i V_t(i)$$

Let $\mu_{i,j}$ be defined as $\mu_{i,j} = p_{i,j}\mu_i$, the derivative of the element j of the distribution vector is

$$\frac{dV_t(j)}{dt} \quad = \quad -\mu_j V_t(j) + \sum_{i\in E, i\neq j} \mu_{i,j} V_t(i)$$

The derivative of the distribution can be written as a matrix:

$$\frac{dV_t}{dt} = V_t A \tag{12}$$

where A is defined as ($a_{i,j} = \mu_{i,j}$ if $i \neq j$ and $a_{j,j} = -\mu_j$):

$$A = \begin{pmatrix} -\mu_0 & \mu_{0,1} & \mu_{0,2} & \cdots & \cdots & \cdots \\ \mu_{1,0} & -\mu_1 & \mu_{1,2} & \mu_{1,3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \mu_{n,n-1} & -\mu_n & \mu_{n,n+1} & \cdots \end{pmatrix}$$

The matrix A is called the infinitesimal generator matrix of the Markov process $(X_t)_{t\in\mathbb{R}^+}$.

The solution of the differential equation $\frac{dV_t}{dt} = V_t A$ is

$$V_t = Ce^{At}$$

where C is a constant ($C = V_0$) and

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + ...$$

Remark 4. *We have seen that the time spent in a state is exponentially distributed. It corresponds to the minimum of several exponential random variables, each one leading to another state. The parameter μ_j is then the sum of the parameters of these exponential random variables, and $\mu_{i,j}$ is the parameter of the exponential r.v. leading from state i to state j. So, we get*

$$\mu_i = \sum_{j\in E} \mu_{i,j}$$

2.3 Equilibrium distribution of a Markov process.

We have seen in the Section on Markov chains that there may exist an equilibrium distribution of V_n. We call this equilibrium distribution π. It corresponds to the asymptotic distribution of V_t when t tends to infinity. This equilibrium distribution does not depend on t. So, if such a distribution π exists for the Markov process, it should verify $\pi A = 0$. Indeed, if t tends to infinity on both sides of equation 12 we get $\frac{d\pi}{dt} = \pi A$ and since π does not depend on t, $\frac{d\pi}{dt} = 0$.

Theorem 3. ***Existence of an equilibrium distribution.*** *Let $(X_t)_{t\in \mathbb{R}^+}$ be a homogeneous, irreducible Markov process. If there exists a solution to equations*

$$\pi A = 0 \textit{ and } \sum_{i\in E} \pi_i = 1 \tag{13}$$

then this solution is unique and

$$\lim_{t\to+\infty} V_t = \pi$$

Theorem 4. ***Ergodicity.*** *If there exists a solution to equations (13) then the Markov process is ergodic, for all $j \in E$*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \mathbb{1}_{X_t^- = j} dt = \pi_j$$

Exercise 14. *We consider the system described in Exercise 13.*

1. *Express the infinitesimal generator.*

2. *What is the existence condition of an equilibrium distribution? Express it when it exists.*

Answers

1. *From remark 4, we know that we can see $a_{i,j}$ $(i \neq j)$ as the parameter of the exponential distribution leading from state i to state j. For instance, if the system is in state j, the exponential random variable leading from i to $i+1$ has parameter λ. So, $a_{i,i+1} = \lambda$. In the same way, the exponential random variable leading from i to $i-1$ has parameter μ. For the other states j with $|i-j| > 1$, the transition probability are nill. We have,*

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & ... \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & ... \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & ... \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & ... \\ ... & ... & ... & ... & ... & ... \end{pmatrix}$$

2. *We try to find a solution to equations 13. If such a solution exists, the equilibrium distribution exists and is given by π. $\pi A = 0$ leads to the following set of equations:*

$$
\begin{aligned}
-\lambda\pi_0 + \mu\pi_1 &= 0 & (14)\\
\lambda\pi_0 - (\lambda+\mu)\pi_1 + \mu\pi_2 &= 0 & (15)\\
&\cdots \\
\lambda\pi_{k-1} - (\lambda+\mu)\pi_k + \mu\pi_{k+1} &= 0 & (16)
\end{aligned}
$$

We express all the π_k with regard to π_0. Equations 14 and 15 lead to $\pi_1 = \frac{\lambda}{\mu}\pi_0$ and $\pi_2 = \frac{\lambda^2}{\mu^2}\pi_0$. We assume that π_k has the following form $\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0$. We prove that this expression of π is a solution of the system by subsituting π_{k-1}, π_k and π_{k+1} in 16. We get,

$$
\begin{aligned}
\lambda\pi_{k-1} - (\lambda+\mu)\pi_k + \mu\pi_{k+1} &= \left(\lambda\left(\frac{\lambda}{\mu}\right)^{k-1} - (\lambda+\mu)\left(\frac{\lambda}{\mu}\right)^k + \mu\left(\frac{\lambda}{\mu}\right)^{k+1}\right)\pi_0 \\
&= \left(\frac{\lambda^k}{\mu^{k-1}} - \frac{\lambda^{k+1}}{\mu^k} - \frac{\lambda^k}{\mu^{k-1}} + \frac{\lambda^{k+1}}{\mu^k}\right)\pi_0 \\
&= 0
\end{aligned}
$$

$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0$ is thus a solution of $\pi A = 0$. We have now to find the value of π_0 such that $\sum_{i=0}^{+\infty} \pi_k = 1$.

$$
\begin{aligned}
\sum_{i=0}^{+\infty} \pi_k &= \pi_0 + \sum_{i=1}^{+\infty} \pi_0 \left(\frac{\lambda}{\mu}\right)^k \\
&= \pi_0\left[1 + \frac{\lambda}{\mu}\sum_{i=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^k\right]
\end{aligned}
$$

If $\frac{\lambda}{\mu} \geq 1$, the sum diverges and there is no solution. If $\frac{\lambda}{\mu} < 1$, the sum converges and we get

$$
\begin{aligned}
\sum_{i=0}^{+\infty} \pi_k &= \pi_0\left[1 + \frac{\lambda}{\mu}\frac{1}{1-\frac{\lambda}{\mu}}\right] \\
&= \pi_0 \frac{1}{1-\frac{\lambda}{\mu}}
\end{aligned}
$$

thus, in order to ensure that $\sum_{i=0}^{+\infty} \pi_k = 1$ *we take* $\pi_0 = 1 - \frac{\lambda}{\mu}$*, and we finally obtain the following equilibrium distribution (for* $\frac{\lambda}{\mu} < 1$*:*

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right)$$

3 Jump chain

3.1 A natural jump chain

There is a natural way to associate a Markov chain with a Markov process. Let $(Y_i)_{i\in\mathbb{N}}$, defined as $Y_0 = X_0, Y_1 = X_{T_1}, ..., Y_n = X_{T_n}$ where T_n are the times X_t changes of states. This Markov chain is called the **jump chain** of the Markov process X_t. Its transition probabilities are $p_{i,j}$. The probability for the jump chain to stay in the same state twice is nil. The equilibrium distribution of a jump chain will in general be different from the distribution of the Markov process generating it. This is because the jump chain ignores the length of time the process remains in each state.

Exercise 15. *We consider a system with one server. When the server is idle, an incoming customer is served with an exponential r.v. with parameter μ. If the server is busy (there is a customer being served), an incoming customer is dropped and never comes back. The interarrival times of the customers are a sequence of independent exponential r.v. with parameter λ.*

1. *Compute the equilibrium distribution of the Markov process describing the number of customers in the system.*

2. *Compute the equilibrium distribution of the jump chain.*

Solution. *The infinitesimal generator is*

$$A = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

From equation $\pi A = 0$, we get $\pi_1 = \frac{\lambda}{\mu}\pi_0$. $\pi_0 + \pi_1 = 1$ leads to $\pi_1 = \frac{\lambda}{\lambda+\mu}$ and $\pi_0 = \frac{\mu}{\lambda+\mu}$.

For the jump chain, the transition matrix is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Equation $\pi P = \pi$ and $\pi_0 + \pi_1 = 1$ leads to $\pi_1 = \frac{1}{2}$ and $\pi_0 = \frac{1}{2}$. So, the stationary distribution of the Markov chain and Markov process are different. The jump chain considers only the fact that when we are in state 0 we have no choice but to go to state 1 and inversely. It does not capture the time spent in each state.

Exercise 16. *The equilibrium distribution of a jump chain may be directly deduced from the distribution of the Markov process. First, we remark that*

$$\pi P = \pi \Leftrightarrow \forall j \in E, \sum_{i\in E} \pi_i p_{i,j} = \pi_j \tag{17}$$

and,

$$\pi A = 0 \Leftrightarrow \forall j \in E, \sum_{i\in E} \pi_i \mu_{i,j} = \pi_j \mu_j \tag{18}$$

Let $(X_t)_{t\in\mathbb{R}^+}$ *be a Markov process possessing an equilibrium distribution* π.

1. *Show that* $\pi_j^{'} = \pi_j \mu_j$ *is a solution of equation* $\pi P = \pi$.
2. *Show that* $\pi_j^{'} = C\pi_j \mu_j$ *is the equilibrium distribution of the jump chain if and only if* $C^{-1} = \sum_{i\in E} \mu_i \pi_i$ *is finite.*

Solution. *The substitution of* $\pi^{'}$ *in equation (17) leads to*

$$\begin{aligned}
\sum_{i\in E} \pi_i^{'} p_{i,j} &= \sum_{i\in E} \pi_i \mu_i p_{i,j} \\
&= \sum_{i\in E} \pi_i \mu_i \frac{\mu_{i,j}}{\mu_i} \text{ see remark 4} \\
&= \sum_{i\in E} \pi_i \mu_{i,j} \\
&= \pi_j \mu_j \text{ from equation 18} \\
&= \pi_j^{'}
\end{aligned}$$

$\pi^{'}$ *verifies* $\pi P = \pi$. *It must be normalized to be an equilibrium distribution. It can be normalized if and only if* $C^{-1} = \sum_{i\in E} \pi_i^{'}$ *is finite. In this case, there exists a unique equilibrium distribution* $\pi^{''} = C\pi'$.

Remark 5. *A closed formula, solutions to equations* πA *(or* $\pi P = \pi$*), can only be found in special cases. For most of the Markov processes (or Markov chains) numerical computations are necessary. A simple way to find a solution when* E *has* N *elements (*N *is supposed finite) is to apply the following algorithm.*

For the Markov chain, it is easy. A vector $\pi^{(0)}$ *is initialized with* $\pi_i^{(0)} = \frac{1}{N}$. *Then* π^n *is computed as* $\pi^{(n)} = \pi^{(n-1)}P$. *We iterate* $\pi^{(n)}$ *until the difference* $\max_{i\in E}\left(\pi^{(n)} - \pi^{(n-1)}\right)$ *is less than a given threshold* ϵ.

For the Markov process, we use a similar algorithm. $\pi A = 0$ *can be written as* $\pi = c\pi A + \pi$ *where* c *is a constant. If we define the matrix* P *as* $cA + I$, *equation* $\pi A = 0$ *becomes* $\pi P = \pi$ *and the algorithm of the Markov chain above can be applied. Note that the constant* c *may be chosen such that* $c < \frac{1}{\max_{i\in E} \mu_i}$.

3.2 Other jump chains

In this Section, we consider Birth and Death processes. With such processes, a process X_t increases or decreases by 1 at each jump. This process is not necessarily Markovian. We associate to these processes two jump chains. Let T_i be the i^{th} instant the Markov process is incremented and S_i the i^{th} instant the process is decremented. The first chain is a sequence of random variables (not necessary Markovian) corresponding to the different states of the process at time T_i^-. The second chain is a sequence of random variables (not necessary Markovian) corresponding to the different states of the process at time S_i^+.

Proposition 4. *The equilibrium distribution (when it exists) of the two chains are equals.*

Proposition 5. *If the random variables $T_{i+1}-T_i$ of the process are independently, identically and exponentially distributed (it is a Poisson point process in this case), the equilibrium distribution (when it exists) of the chain at the arrival time is the same as the equilibrium distribution of the process.*

Proof. Let $p_a(n,t)$ be the stationary probability at an arrival time ($\mathbb{P}(X_{T_j^-} = n)$), and N_t the number of arrivals in $[0,t)$,

$$\begin{aligned}
p_a(n,t) &= \lim_{dt\to 0} \mathbb{P}\left(X_t = n | N_{t+dt} - N_t = 1\right) \\
&= \lim_{dt\to 0} \frac{\mathbb{P}\left(X_t = n, N_{t+dt} - N_t = 1\right)}{\mathbb{P}\left(N_{t+dt} - N_t = 1\right)} \\
&= \lim_{dt\to 0} \frac{\mathbb{P}\left(N_{t+dt} - N_t = 1 | X_t = n\right)\mathbb{P}\left(X_t = n\right)}{\mathbb{P}\left(N_{t+dt} - N_t = 1\right)}
\end{aligned}$$

Since arrivals are modeled by a Poisson point process, the probability that there is an arrival between t and $t + dt$ is independent of the value of X_t. So, we get

$$\begin{aligned}
p_a(n,t) &= \lim_{dt\to 0} \frac{\mathbb{P}\left(N_{t+dt} - N_t = 1\right)\mathbb{P}\left(X_t = n\right)}{\mathbb{P}\left(N_{t+dt} - N_t = 1\right)} \\
p_a(n,t) &= \lim_{dt\to 0} \mathbb{P}\left(X_t = n\right) \\
p_a(n,t) &= \mathbb{P}\left(X_t = n\right)
\end{aligned}$$

□

4 Queuing Theory

Queueing theory is the mathematical study of waiting lines (or queues). The theory enables mathematical analysis of several related processes, including arriving at the queue, waiting in the queue , and being served by the server(s) at the front of the queue. The theory permits the derivation and calculation of several performance measures including the average waiting time in the queue or the system, the expected number of customers waiting or receiving service and the probability of encountering the system in certain states, such as empty, full, having an available server or having to wait a certain time to be served.

4.1 Kendall Notation

A queue is described in shorthand notation by $A/B/C/D/E$ or the more concise A/B/C. In this concise version, it is assumed that $D = +\infty$ and $E = FIFO$.

1. A describes the arrival process of the customers. The codes used are:
   (a) M (Markovian): interarrival of customers are independently, identically and exponentially distributed. It corresponds to a Poisson point process.
   (b) D (Degenerate): interarrival of customers are constant and always the same.
   (c) GI (General Independent): interarrival of customers have a general distribution (there is no assumption on the distribution but they are independently and identically distributed).
   (d) G (General): interarrival of customers have a general distribution and can be dependent on each other.
2. B describes the distribution of service time of a customer. The codes are the same as A.
3. C is the number of servers.
4. D is the number of places in the system (in the queue). It is the maximum number of customers allowed in the system including those in service. When the number is at its maximum, further arrivals are turned away (dropped). If this number is omitted, the capacity is assumed to be unlimited, or infinite.
5. E is the service discipline. It is the way the customers are ordered to be served. The codes used are:

(a) $FIFO$ (First In/First out), the customers are served in the order they arrived in.

(b) $LIFO$ (Last In/First out), the customers are served in the reverse order to the order they arrived in.

(c) $SIRO$ (Served In Random Order), the customers are served randomly.

(d) $PNPN$ (Priority service), the customers are served with regard to their priority. All the customers of the highest priority are served first, then the customers of lower priority are served, and so on. The service may be preemptive or not.

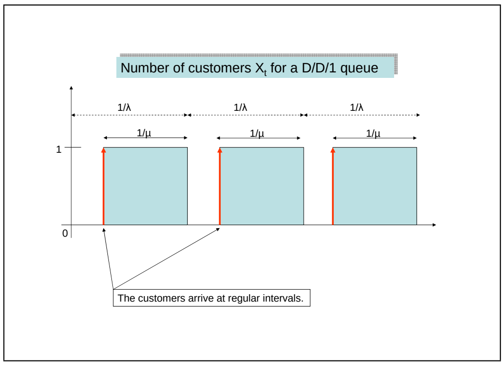
(e) PS (Processor Sharing), the customers are served equally. System capacity is shared between customers and they all effectively experience the same delay.

Exercise 17. *What is the Kendall notation for the queue of exercise 13.*

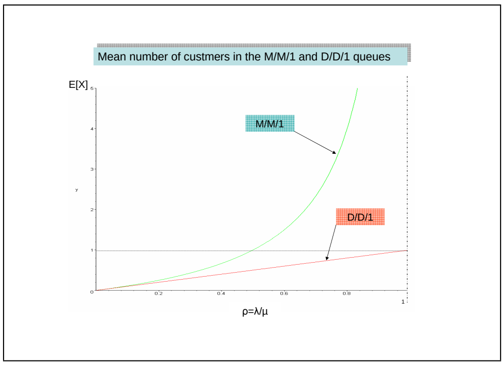
4.2 M/M/1

Exercise 18. *M/M/1 Let there be a M/M/1 queue.*

1. *What is the average time between two successive arrivals?*
2. *What is the average number of customers coming per second ?*
3. *What is the average service time of a customer ?*
4. *What is the average number of customers that the server can serve per second ?*
5. *Compute the equilibrium distribution and give the existence condition of this distribution.*
6. *Compute the mean number of customers in the queue under the equilibrium distribution.*
7. *Compute the average response time (time between the arrival and the departure of a customer).*
8. *Compute the average waiting time (time between the arrival and the beginning of the service).*
9. *Compare the obtained results with results of a D/D/1 queue with the same arrival and service rates.*
10. *Plot the mean number of customers and the mean response times for the two queue (M/M/1 and D/D/1) for $\frac{\lambda}{\mu}$ varying from 0 to 1.*



(a) The number of customers in the D/D/1 queue



(b) Mean number of customers in the D/D/1 and M/M/1 queues

Figure 1: Exercise 18.

Answers

1. *The mean time between two arrivals is the mean of an exponential random variable with parameter* λ*, thus* $\frac{1}{\lambda}$.

2. *The mean number of customers coming in the system is then* λ.

3. *The mean service time is* $\frac{1}{\mu}$.

4. *The mean number of customers that can be served by the server is then* μ.

5. *According to exercise 14, the stationnary distribution exists when* $\rho = \frac{\lambda}{\mu} < 1$ *and is equal to* $\pi_k = \rho^k(1-\rho)$.

6. *The mean number of customers in the queue under the stationnary distribution is by definition:*

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=0}^{+\infty} k\pi_k = \sum_{i=0}^{+\infty} k\rho^k(1-\rho) \\
&= (1-\rho)\rho\sum_{i=0}^{+\infty} k\rho^{k-1} = (1-\rho)\rho\sum_{i=0}^{+\infty}\frac{d}{d\rho}\rho^k \\
&= (1-\rho)\rho\frac{d}{d\rho}\left(\sum_{i=0}^{+\infty}\rho^k\right) = (1-\rho)\rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \\
&= (1-\rho)\rho\frac{1}{(1-\rho)^2} = \rho\frac{1}{(1-\rho)}
\end{aligned}$$

7. *The response time* R *is defined as the time between the arrival of a customer in the system and its departure. It corresponds to the time*

spent in the system (time to queue and time in the server). To compute the mean response time, we use the result of exercise 26: $\mathbb{E}[X] = \lambda\mathbb{E}[R]$*. For the M/M/1 queue, the result is thus*

$$\mathbb{E}[R] = \frac{\mathbb{E}[X]}{\lambda} = \frac{\rho}{\lambda(1-\rho)}$$

8. *The waiting time* W *is defined as the time that the customer spent in the system before being served. The mean waiting time is thus the difference between the mean response time and the mean service time:*

$$\mathbb{E}[W] = \mathbb{E}[R] - \frac{1}{\mu}$$

9. *The D/D/1 queue is a queue where both interarrivals and service time are constant. Since, the arrival rate and service time are the same in average, we suppose that a server will serve a customer in a time equal to* $\frac{1}{\mu}$ *and that the customers arrive at regular interval equal to* $\frac{1}{\lambda}$*. If we suppose that* $\lambda < \mu$*, the service time is less than the interarrival. The number of customers varies from* 0 *to* 1 *as shown in Figure 1(a). The probability* π_1 *that there is one customer in the system is then the ratio* $\frac{\frac{1}{\mu}}{\frac{1}{\lambda}} = \rho$ *(with* $\rho = \frac{\lambda}{\mu}$*) and the probability* π_0 *that there is no customer is* $1-\rho$*. The mean number of customers is then* $\mathbb{E}[X] = 0\pi_0 + 1\pi_1 = \rho$ *and* $\mathbb{E}[R] = \frac{1}{\mu}$*.*

10. *In Figure 1(b), we plot the mean number of customers* $\mathbb{E}[X]$ *for the two queues when* ρ *varies from* 0 *to* 1*. For the D/D/1 queue,* $\mathbb{E}[X]$ *tend to* 1*. For the M/M/1 queue,* $\mathbb{E}[X]$ *tends to infinity as* $\rho \to 1$*. So, for the second case, even if the arrival rate is less than service rate, the randomness of the interarrivals and service times leads to an explosion of the number of customers when the system is close to saturation.*

4.3 M/M/K

Exercise 19. *M/M/K Let there be a M/M/K queue.*

1. *Compute the equilibrium distribution and give the existence condition of this distribution.*

2. *Compute the mean number of customers in the queue under the equilibrium distribution.*

3. *Compute the average response time.*

4. *Compute the average waiting time.*

4.4 M/M/$+\infty$

Exercise 20. *M/M/$+\infty$ Let there be a M/M/$+\infty$ queue.*

1. *Compute the equilibrium distribution and give the existence condition of this distribution.*
2. *Compute the mean number of customers in the queue under the equilibrium distribution.*
3. *Compute the average response time.*
4. *Compute the average waiting time.*

4.5 M/M/K/K+C

Exercise 21. *M/M/K/K+C Let there be a M/M/K/K+C queue.*

1. *Compute the equilibrium distribution and give the existence condition of this distribution.*
2. *Compute the mean number of customers in the queue under the equilibrium distribution.*
3. *Compute the average response time.*
4. *Compute the average waiting time.*

Solution. *It is of course an irreducible, homogeneous Markov process. Firstly, we compute the infinitesimal generator A and we try to solve $\pi A = 0$ and $\sum_{i=0}^{K+C} \pi_i = 1$. Since E has a finite number of elements, there is always a solution to these two equations.*

$$
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & ... & \\
\mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & ... & \\
0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & 0 & ... & \\
0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & 0 & ... & \\
 & & ... & & & & & \\
 & & ... & & & & & \\
... & (K-1)\mu & -((K-1)\mu+\lambda) & \lambda & 0 & ... & & \\
... & 0 & K\mu & -(K\mu+\lambda) & \lambda & ... & & \\
... & ... & 0 & K\mu & -(K\mu+\lambda) & \lambda & 0 & ... \\
 & & ... & & & & & \\
 & & ... & & & & & \\
... & & & ... & 0 & 0 & \mu K & -\mu K
\end{pmatrix}
$$

πA leads to:

$$
\begin{aligned}
-\lambda\pi_0 + mu\pi_1 &= 0 \quad (19)\\
\lambda\pi_0 + -(\lambda+\mu)\pi_1 + 2\mu\pi_2 &= 0 \quad (20)\\
\lambda\pi_1 + -(\lambda+2\mu)\pi_2 + 3\mu\pi_3 &= 0 \quad (21)\\
&\cdots\\
\lambda\pi_{k-1} + -(\lambda+k\mu)\pi_k + (k+1)\mu\pi_{k+1} &= 0 \text{ if } k < K \quad (22)\\
&\cdots\\
\lambda\pi_{k-1} + -(\lambda+(K+C)\mu)\pi_k + (K+C)\mu\pi_{k+1} &= 0\\
\text{if } K \le k \le K+C & \quad (23)
\end{aligned}
$$

From equation 19, we get $\pi_1 = \frac{\lambda}{\mu}\pi_0 = \rho\pi_0$. *From equation 20, we get* $\pi_2 = \frac{\rho^2}{2}\pi_0$. *From equation 21, we get* $\pi_3 = \frac{\rho^3}{3!}\pi_0$. *For* $k < K$, *we assume that* $\pi_k = \frac{\rho^k}{k!}\pi_0$. *It is verified by equation 22. For* $K \le k \le K+C$, *from equation 23 we get* $\pi_k = \frac{1}{K^{k-K}}\frac{\rho^k}{K!}\pi_0$. *From equation* $\sum_{i=0}^{K+C}\pi_i = 1$, *we get*

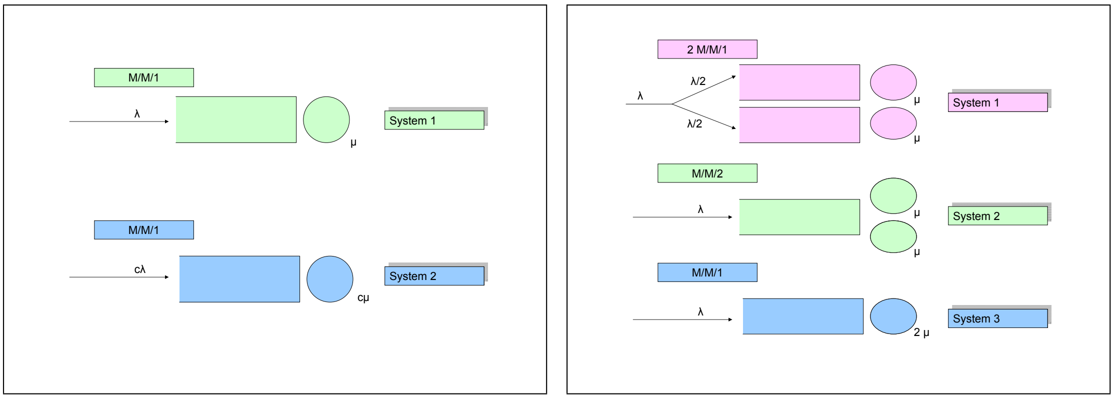
$$
\begin{aligned}
\pi_0 &= \left[\sum_{i=0}^{K-1}\frac{\rho^i}{i!} + \sum_{i=K}^{K+C}\frac{\rho^i}{K!K^{i-K}}\right]^{-1}\\
&= \left[\sum_{i=0}^{K-1}\frac{\rho^i}{i!} + \frac{\rho^K}{K!}\sum_{i=K}^{C+K}\frac{\rho^{i-K}}{K^{i-K}}\right]^{-1}\\
&= \left[\sum_{i=0}^{K-1}\frac{\rho^i}{i!} + \frac{\rho^K}{K!}\sum_{i=0}^{C}\frac{\rho^i}{K^i}\right]^{-1}\\
&= \left[\sum_{i=0}^{K-1}\frac{\rho^i}{i!} + \frac{\rho^K}{K!}\frac{1-\frac{\rho^{C+1}}{K^{C+1}}}{1-\frac{\rho}{K}}\right]^{-1}
\end{aligned}
$$

For this queue there is no existence condition of the equilibrium distribution since the number of states is finite ($i = 0, .., K+C$).

The results for the other queue can be easily deduced from the formula above. For instance, the equilibrium distribution for the M/M/K queue is obtained as the limit of above equation when C tends to $+\infty$ *with the condition that* $\frac{\rho}{K} < 1$.

4.6 Some interesting results

Exercise 22. *We try to compare two networks as represented in Figure 2(a). In the first one, packets are big and in the second network packets are c times smaller. Therefore, for a constant number of bytes to transport, the number of packets network arriving in a node in the second network is c times greater. We assume that the number of bytes that the servers can process is the same*



(a) Exercise 22 (b) Exercise 23

Figure 2: Comparison of different queues

for the two networks. So, servers in network B can process c times as many packets.

The mathematical model is as follows We consider two M/M/1 queues called A and B corresponding to each network. The arrival rate of A is λ, and it is λc (with $c > 1$) for B. The service rate is μ for queue A, and μc for queue B.

1. *Compare the mean number of customers in the two queues.*
2. *Compare the average response time in the two queues.*
3. *If the average packet size in network A is L bytes, and $\frac{L}{c}$ bytes in network B, compute the mean number of bytes that the packets occupy in the node.*
4. *Conclude.*

Solution. *For queue A ($\rho = \frac{\lambda}{\mu}$):*

$$\mathbb{E}[X] = \frac{\rho}{1-\rho} \text{ and } \mathbb{E}[R] = \frac{1}{\mu}\frac{\rho}{1-\rho}$$

For queue B:

$$\mathbb{E}[X] = \frac{\rho}{1-\rho} \text{ and } \mathbb{E}[R] = \frac{1}{c\mu}\frac{\rho}{1-\rho}$$

So, in the second case (B) where packets are c times smaller, the mean number of packets in the queue are the same. It is however more efficient because these packets are c times smaller and thus will take c times less memory in the node. The response time is then also shorter (since the *server capacity is constant), and is c times smaller than in queue A. The reason for this difference, is variance. The variance increases with packet size (for exponential law).*

Exercise 23. *We try to compare three systems. Arrival rates and service rates are the same for the three systems. The only difference is that for the first system there are two queues, in the second system there is only one queue but two servers and for the third system there is one queue and one server but the server is two times faster. The mathematical model is as follows. We consider two independent M/M/1 queues to model the first system. The arrival rate for each queue is* $\frac{\lambda}{2}$ *and the service rate of each server is* μ*. For the second system, we consider a M/M/2 queue with arrival rate* λ *and service rate* μ *for each server. We suppose that* $\frac{\rho}{2} = \frac{\lambda}{2\mu} < 1$*. For the third system, we consider a M/M/1 queue with arrival rate* λ *and service rate* 2μ*.*

The three systems are represented in Figure 2(b). We suppose that $\frac{\rho}{2} = \frac{\lambda}{2\mu} < 1$*.*

1. *What is the load for the three systems?*
2. *What is the mean number of customers for the two systems?*
3. *What is the average response time for the two systems?*
4. *Conclude.*

Solution.

The loads for the three systems are the same: $\frac{\lambda}{2\mu}$.
The mean number of customers for the two M/M/1 queues (system 1) is

$$\mathbb{E}[X_1] = 2\frac{\frac{\rho}{2}}{1-\frac{\rho}{2}} = \frac{\rho}{1-\frac{\rho}{2}}$$

For the queue M/M/2 (system 2), it is more complex. In equations on M/M/K/K+C queue, we take $C = +\infty$ *and* $K = 2$*, we obtain:*

$$\begin{aligned} \pi_k &= \frac{\rho^k}{2^{k-1}}\pi_0 \text{ for } k \geq 1 \\ \pi_0 &= \frac{1-\frac{\rho}{2}}{1+\frac{\rho}{2}} \end{aligned}$$

Now, we can compute the mean number of customers in the queue:

$$\begin{aligned} \mathbb{E}[X_2] &= \sum_{k=0}^{+\infty} k\pi_k \\ &= \rho\sum_{k=1}^{+\infty} k\left(\frac{\rho}{2}\right)^{k-1}\pi_0 \end{aligned}$$

If x is real $\in]0,1[$, we get (to prove that, we have to consider that kx^{k-1} is the derivative of x^k):

$$\sum_{k=1}^{+\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

With $x = \frac{\rho}{2}$, we get:

$$\begin{aligned}
\mathbb{E}[X_2] &= \rho \sum_{k=1}^{+\infty} k \left(\frac{\rho}{2}\right)^{k-1} \pi_0 \\
&= \frac{\rho}{(1-\frac{\rho}{2})^2} \pi_0 \\
&= \frac{\rho}{(1-\frac{\rho}{2})^2} \frac{1-\frac{\rho}{2}}{1+\frac{\rho}{2}} \\
&= \frac{\rho}{(1-\frac{\rho}{2})(1+\frac{\rho}{2})}
\end{aligned}$$

For the third system (M/M/1), we obtain:

$$\mathbb{E}[X_3] = \frac{\frac{\rho}{2}}{1-\frac{\rho}{2}}$$

It appears that the mean number of customers is greater in system 1 and 2: $\mathbb{E}[X_1] > \mathbb{E}[X_2] > \mathbb{E}[X_3]$. It is due to the fact that in system 1, with the two M/M/1 queues, when one server is empty and the other queue has some customers, the first one is not use leading to the waste of a part of the capacity. Moreover, the variance of the interarrival and service time is really smaller for system 3 leading to better performances.

4.7 M/GI/1 and GI/M/1

When interarrival or service time is not exponential, the process describing the number of customers is no more Markovian. The jump chain taken at each jump of the process is not a Markov chain. However, for M/GI/1 and GI/M/1 we can build a Markov chain from the Markov process. These jump chains are those defined in Section 3.2.

Exercise 24. *1. Prove that the jump chain taken at the arrival time of a GI/M/1 is a Markov chain.*

2. Prove that the jump chain taken at the departure time of a M/GI/1 is a Markov chain.

Solution. *For the M/GI/1, we have to prove that*

$$\mathbb{P}(X_n = j | X_{n-1} = i_{n-1}, ..., X_0 = i_0) = \mathbb{P}\left(X_{S_n^-} = j | X_{S_{n-1}^-} = i_{n-1}, ..., X_{S_0^-} = i_0\right)$$

Since the interarrival process is a sequence of i.i.d. exponential r.v., it is memory less. Thus the number of customers coming between S_{n-1}^- and S_n^- is independent of the past. Moreover, there is only one service between S_{n-1}^- and S_n^- (it is not cut), its duration is independent of the past since service times are independently distributed. Therefore, the probability that the chain changes from state i to state j depends only on i and j. Therefore, it is a Markov chain.

For the M/GI/1 and GI/M/1 queues, the transition probabilities depend on the considered "GI" distribution. For instance, for M/GI/1 queue, from state j there is a positive probability to go to state $j-1$ if no customer arrives between the two departures (between X_{n-1} and X_n) and positive probabilities to go to state $j+k$ if k customers arrive during the service of the customer currently in the server (between X_{n-1} and X_n).

Exercise 25. *We consider the M/D/1 and D/M/1 queues.*

1. *Compute the transition matrix of the jump chains for these two queues.*
2. *Compute the equilibrium distribution for the two queues. What are the existence conditions?*
3. *Compute the mean number of customers and the mean response time.*

Proposition 6. *Khintchin-Pollazcek formula. The mean number of customers X in a M/GI/1 queue is*

$$\mathbb{E}[X] = \rho + \frac{\rho^2 \left(1 + \frac{var(Y)}{\mathbb{E}[Y]^2}\right)}{2(1-\rho)}$$

where Y is a random variable with the same distribution as the service time of a customer and $\rho = \frac{\lambda}{\mu}$.

5 Miscellaneous

Exercise 26. *We consider two moving walkways. The first one has a constant speed equal to 11km/h and the second 3km/h. Both moving walkways have a length of 500 meters. We also assume that there is a person arriving on each moving walkway each second.*

- *Give the number of persons on each walkway.*
- *Deduce a formula which links the number of customers in a system, the mean arrival rate and the response time.*